Approximation Spaces in the Numerical Analysis of Cauchy Singular Integral Equations

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Dr. rer. nat. Uwe Luther

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Gutachter: Prof. Dr. Peter Junghanns (Chemnitz)
Prof. Dr. Jürgen Prestin (Lübeck)
Prof. Dr. Maria Rosaria Capobianco (Neapel)

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Introduction

The present paper is devoted to the numerical analysis of a quadrature method for the singular integral equation

$$a(x)\,\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{b(t)\,\phi(t)}{t-x}\,dt + \frac{1}{\pi}\int_{-1}^{1} k(x,t)\,\phi(t)\,dt = g(x)\,, \quad x \in (-1,1)\,. \tag{0.1}$$

Here the functions a, b, k, g are given and ϕ is looked for. The first integral on the left hand side of (0.1) is understood in the sense of the Cauchy principle value, while the second one is a usual Lebesgue integral. We assume that a and b are real-valued and Hölder continuous on $[-1,1]$, that $a^2 + b^2 > 0$, and that there exist certain weight functions u and w with $(uw)^{-1} \in \mathbf{L}^1(-1,1)$ such that $u(x)\,g(x) = h(x)$ and $u(x)\,k(x,t)\,w(t) = h_1(x,t) + u(x)\,(t-x)^{-1}[h_2(x,t) - h_2(x,x)]\,w(t)$ with Hölder continuous functions h and h_i on $[-1,1]$ and $[-1,1]^2$, respectively. The weight functions are taken from the class of power weights which are of the form

$$u(x) = \prod_{i=1}^{N} |x - x_i|^{\alpha_i}\,, \qquad -1 \le x_1 < \ldots < x_N \le 1\,, \quad 0 < \alpha_i < 1\,.$$

(For $N = 0$ we set $u = 1$.) There are several applications in which equations of type (0.1) appear. Here we refer to the literature, for example [Mu, Mo, K, M2]; see also the chapters on applications in [M1, MP] and the references given therein. Since one is often interested in function values of the solution of equation (0.1) and since usually (0.1) is not explicitly solvable, pointwise error estimates for numerical methods are of great interest. Here we will investigate the weighted uniform convergence of a certain approximation method, i.e., the convergence in the norm

$$\|f\|_u = \max_{x \in [-1,1]} |u(x) f(x)|\,. \tag{0.2}$$

This is motivated by the assumed property $ug \in \mathbf{C}[-1,1]$ of the right hand side of (0.1). It is well known that the main part A of the operator defined by the left hand side of (0.1),

$$(A\phi)(x) = a(x)\,\phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{b(t)\,\phi(t)}{t-x}\,dt\,,$$

is, in general (i.e. if $b \not\equiv 0$), not bounded in the weighted space of continuous functions

$$\mathbf{C}_u = \left\{ f \in \mathbf{C}\big([-1,1] \setminus \{x_i\}_{i=1}^N\big) : \, uf \in \mathbf{C}[-1,1] \text{ (continuous extension)} \right\}$$

which corresponds to the norm (0.2). For this reason one cannot study (0.1) as an equation in $\mathbf{C}_u$. One possibility to tackle this problem is to consider (0.1) in certain subspaces of $\mathbf{C}_u$ in which A is continuous. For example, it is known that (see [GK, Section 9.10])

$$A \text{ is bounded in } \mathbf{H}^\eta_u \quad \text{if} \quad a, b \in \mathbf{H}^\eta\,, \ \{-1,1\} \subseteq \{x_i\}\,, \ \text{and} \ \alpha_i > \eta \ \text{for all } i\,,$$

where $\mathbf{H}^\eta = \big\{ f \in \mathbf{C}[-1,1] : f \text{ is Hölder continuous with exponent } \eta \big\}$, $\eta \in (0,1)$, and $\mathbf{H}^\eta_u = \big\{ f : \, fu \in \mathbf{H}^\eta, \ (fu)(x_i) = 0 \text{ for all } i \big\}$ (endowed with the Hölder norm of fu). But here we will proceed in another way for the following reason: We are not only interested in the $\mathbf{C}_u$–convergence to ϕ of approximate solutions ϕ_n. We also want to obtain an error estimate of the form $\|\phi - \phi_n\|_u = O(n^{\varepsilon-\gamma})$, $\varepsilon > 0$ arbitrary, supposed that, in the norm of $\mathbf{C}_u$, g can be approximated with order $O(n^{-\gamma})$ by polynomials $g_n(x) \in \operatorname{span}\{x^k\}_{k=0}^n$. If we consider (0.1) in a space smaller than $\mathbf{C}_u$ and if we study convergence in the corresponding stronger norm, then we cannot hope that a convergence order $O(n^{\varepsilon-\gamma})$ ($\varepsilon > 0$ arbitrary) holds true in this norm, since a loss of some power of n must be expected which corresponds, in some sense, to the "degree of strongness" of the norm. (For example, if we consider $\mathbf{H}_u^\eta$, then η is this degree.)

Instead of studying (0.1) we will investigate a closely related so-called regularized equation

$$\big(I + H + \widehat{A}K\big)\phi = \widehat{A}g\,, \tag{0.3}$$

where $(K\phi)(x) = \pi^{-1}\int_{-1}^1 k(x,t)\,\phi(t)\,dt$ and $\widehat{A}$ is an appropriate left regularizer of A, i.e., an operator with the property $\widehat{A}Af = f + Hf$, $f \in \bigcup_{\eta>0}\mathbf{H}_u^\eta$, with some compact operator H on $\mathbf{C}_u$. If $\phi \in \bigcup_{\eta>0}\mathbf{H}_u^\eta$ solves (0.1), then ϕ is also a solution of (0.3). The advantage of this regularization is that, in contrary to the original equation, (0.3) can be viewed as an operator equation in $\mathbf{C}_u$. For the construction of an approximation method for (0.3) we use the facts that $\widehat{A}$ maps $\bigcup_{\eta>0}\mathbf{H}_u^\eta$ into $v^{\alpha,\beta}\bigcup_{\eta>0}\mathbf{H}_u^\eta$, where

$$v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta\,, \quad \alpha,\beta \in (-1,1)\colon \text{certain numbers depending on } a,b,u$$

and that there exists some Hölder continuous function $h : [-1,1] \to (0,\infty)$ such that the action of the operator $\widehat{\mathcal{A}} = \sigma_{\alpha,\beta}^{-1}\widehat{A}$ with $\sigma_{\alpha,\beta} = v^{\alpha,\beta}h$ is dominated by an operator which maps any polynomial p into the product of another polynomial $q = q(p)$ and some Hölder continuous function depending on a and b the smoothness of which is as higher as smoother a and b. For these two reasons we write (0.3) in the equivalent form

$$f = \sigma_{\alpha,\beta}^{-1}\phi \in \mathbf{C}_{v^{\alpha,\beta}u}\colon \big(I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K}\big)f = \widehat{\mathcal{A}}g\,, \quad \mathcal{H} = \sigma_{\alpha,\beta}^{-1}H\sigma_{\alpha,\beta}I,\ \mathcal{K} = K\sigma_{\alpha,\beta}I$$

and investigate the $\mathbf{C}_{v^{\alpha,\beta}u}$–convergence of a so-called quadrature method

$$f_n \in \operatorname{im} P_n\colon \big(I + P_n\mathcal{H}_n + P_n\widehat{\mathcal{A}}L_n\mathcal{K}_n\big)f_n = P_n\widehat{\mathcal{A}}L_ng\,, \tag{0.4}$$

where L_n and P_n are certain Lagrangian interpolation operators and $\mathcal{H}_n$ and $\mathcal{K}_n$ are approximations of $\mathcal{H}$ and $\mathcal{K}$, respectively, which are obtained by replacing the integrals $(\mathcal{H}f)(x)$ and $(\mathcal{K}f)(x)$ by quadrature rules based on the weight $\sigma_{\alpha,\beta}$.

The idea of proving weighted uniform convergence of polynomial approximation methods for Cauchy singular integral equations via the investigation of regularized equations goes back to Capobianco [C] who considered the collocation method (i.e., $\mathcal{K}_n = \mathcal{K}$) for equations with constant coefficients a,b (which implies $\mathcal{H} = 0$) and regular perturbation K (i.e., continuous $k(x,t)$). The (weighted) uniform convergence of collocation type methods for equations with constant coefficients and weakly singular perturbation is studied in [CR, MoP, Cu1, Cu2]. Generalizations of Capobianco's [C] results to the quadrature method and to the case of variable coefficients can be found in [CJLM, JL1, JL2]. A big number of almost optimal results for the weighted uniform convergence of several approximation methods (collocation, quadrature, product integration, fast algorithms) for equations with variable coefficients and weakly singular perturbation can be found in [L1].

By almost optimal we mean that error estimates of the type $\|\phi - \phi_n\|_u = O(n^{-\gamma} \ln^{k-\delta} n)$ are proved under the assumption that g can be approximated of order $O(n^{-\gamma} \ln^{-\delta} n)$ (in the norm of $\mathbf{C}_u$) by polynomials g_n of degree less than n.

In all mentioned papers only Jacobi weights $u = v^{\rho,\tau}$ are considered and the regularizer $\widehat{\mathcal{A}}$ is a one-sided inverse of $\mathcal{A} = A\sigma_{\alpha,\beta}I$ which is well known from the weighted $\mathbf{L}^2$–theory of Cauchy singular integral equations on $[-1, 1]$ (see, e.g., [E1, E2, JS1, JS2, JS3, MaP, BHS1, BHS2, J, PS]). In particular, in the part h of $\sigma_{\alpha,\beta}$ a parameter integral appears which usually cannot be computed explicitly if the coefficients a and b of A are not constant. From the numerical point of view this means that the methods considered in the above papers are only applicable in the case of constant coefficients a,b and some further special cases.

In the present paper we will use another regularizer $\widehat{A}$ of A and another weight $\sigma_{\alpha,\beta}$ which behaves similar to the weight considered in the classical methods but which contains no complicated part. We construct $\widehat{A}$ with the help of an explicitly known one sided inverse of another weighted operator $A_{\alpha,\beta}$ the coefficients of which are simple trigonometric functions related to the characteristical pair (α, β) belonging to A and u. Although we do not obtain a one-sided inverse of A in this way, we get the equality $\widehat{A}Af = f + Hf$ with a more or less simple weakly singular integral operator H. The quadrature method (0.4) for the resulting regularized equation (0.3) can be implemented with acceptable numerical effort. We prove error estimates for the $\mathbf{C}_{v^{\alpha,\beta}u}$–convergence of this method which are almost optimal if $b(-1)b(1) \neq 0$, at least up to a certain order $\gamma < 2$ which depends on the smoothness of the coefficients a and b of A. This will be done on the basis of a general theory of so-called approximation spaces and their application in the numerical analysis of operator equations. An approximation space consists of elements of some Banach space $\mathbf{X}$ for which the sequences of errors of best approximation by elements of certain subspaces $\mathbf{X}_n$ converge to zero with some prescribed order of convergence. The theory of such spaces was developed, among others, by Pietsch [P], Brudnyi and Krugljak [BK, Section 4.3.C], Almira and Luther [AL1, AL2, AL3, L2]. Here we will give a simplified presentation of this theory in a more restrictive framework as that which is considered, for example, in [AL1]. It is well known that approximation spaces are a powerful tool in the numerical analysis of operator equations. For example, in [JL1, JL2, L1] they are used for the investigation of weighted uniform convergence of polynomial approximation methods for Cauchy singular integral equations. A general stability and convergence theory of approximation methods for operator equations which is based on approximation spaces is given in [L4]. A much more general approach which is applicable to (0.4) will be given in the present paper.

Let us mention that also another powerful tool, namely Banach algebra techniques (see, e.g., [HRS]), is applicable to the numerical analysis of Cauchy singular integral equations on $[-1, 1]$, where non-classical collocation methods for (0.1) can be considered in which only Jacobi weights $v^{\mu,\nu}$ are involved. Corresponding results can be found, for example, in [JW1, JW2, W, JRS, R, JRo, JR]. In these papers necessary and sufficient conditions for the weighted $\mathbf{L}^2$–convergence are proved, even for the case of equations with piecewise continuous coefficients a and b. Compared with these results the advantage of our approach is, besides the possible investigation of weighted uniform convergence instead of weighted $\mathbf{L}^2$–convergence, the fact that the application of approximation spaces automatically yields almost optimal error estimates in the above described sense. For the methods investigated by Banach algebra techniques corresponding results on the order of pointwise convergence are not given in the literature (as far as we know).

The paper is organized as follows. In Chapter 1 we present some results from the theory of approximation spaces. In particular we show, in a general framework, that approximation spaces based on $(\mathbf{X}, \{\mathbf{X}_n\})$ can be used to handle unbounded operators defined on $\bigcup \mathbf{X}_n$. This is the basis of our later investigations of the mapping properties of Cauchy singular integral operators. Moreover, we give characterizations of those approximation spaces based on $\mathbf{C}_u$ which are of main interest in our applications to integral equations. In Chapter 2 we study the mapping properties of the above operator A in spaces of functions with a finite number of weak singularities. First we consider even more general operators with piecewise continuous coefficients in the mentioned approximation spaces based on $\mathbf{C}_u$. Then we show that the well known one-sided $\mathbf{L}^2$–inverse of A can be also considered in certain spaces of locally Hölder continuous functions. Finally we use these results to construct the above mentioned left regularizer $\widehat{A}$ of A. Chapter 3 is devoted to the investigation of weakly singular integral operators with kernels $k(x,t)$ of the above type. We show, roughly spoken, that these operators map $\mathbf{C}_u$ into approximation spaces based on $\mathbf{C}_u$ if $k(x,t)$ as a function in x belongs to an approximation space of the same type. In Chapter 4 we develop the above mentioned general stability and convergence theory of approximation methods for operator equations, where even the concept of regularization is considered from a general point of view. The application of this theory to the investigation of weighted uniform convergence of the quadrature method (0.4) is given in the first part of Chapter 5. Here we use the mapping properties proved in Chapters 2 and 3. In the second part of Chapter 5 we explain how the method (0.4) can be implemented. We end with the presentation of numerical examples in which the computational results are compared with the theoretical results and with the numerical results which are obtained with the help of methods which are already known from the literature.

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Chapter 1

Approximation spaces and unbounded operators

If the solution f of an operator equation $Af = g$ (A an invertible operator on a Banach space $\mathbf{X}$) is approximated by elements $f_n \in \mathbf{X}_n$ of certain subspaces $\mathbf{X}_n$ of $\mathbf{X}$, then it is clear that the error $\|f - f_n\|$ cannot be smaller than the best possible approximation error $E_n(f) = \inf\{\|f - g\| : g \in \mathbf{X}_n\}$. If $E_n(f)$ converges very slowly to zero, then this means that we cannot expect good error estimates if we approximate the solution of $Af = g$ by ansatz functions belonging to $\mathbf{X}_n$. So it is of great interest to know for which right hand sides g the solution $f = A^{-1}g$ belongs to a so-called approximation space

$$\left\{ f \in \mathbf{X} : \sup_n \alpha_n E_n(f) < \infty \right\} \quad (\alpha_n > 0 \text{ given}), \tag{1.1}$$

where the numbers α_n converge to infinity fast enough (e.g., $\alpha_n = n^\gamma$ with some fixed $\gamma > 0$). In other words, we want to know which functions g belong to the image of A restricted on the above approximation space. In the present paper we deal with weighted uniform convergence of approximation methods for certain integral equations. Hence, we are interested in mapping properties of integral operators in approximation spaces of continuous functions. Not all of these operators are bounded in the underlying space $\mathbf{X}$. For this reason we will develop a theory which allows to handle such unbounded operators. In this theory we have to admit spaces (1.1) for sequences $\{\alpha_n\}$ which are not of the classical form $\alpha_n = n^\gamma$. Moreover, it is necessary to consider also approximation spaces of the following type:

$$\left\{ f \in \mathbf{X} : \sum_n \alpha_n E_n(f) < \infty \right\}. \tag{1.2}$$

To have a common approach to both types of spaces (1.1) and (1.2), we present some results from the general theory of approximation spaces.

1.1 Results from the theory of approximation spaces

Let $(\mathbf{X}, \|\,.\,\|)$ be a Banach space and let $\{\mathbf{X}_n\}_{n=1}^\infty$ be a sequence of linear subspaces of $\mathbf{X}$ such that

$$\dim \mathbf{X}_n < \infty \text{ for all } n \quad \text{and} \quad \mathbf{X}_1 \subseteq \mathbf{X}_2 \subseteq \mathbf{X}_3 \subseteq \ldots .$$

The best approximation errors of an element $f \in \mathbf{X}$ (with respect to the subspaces $\mathbf{X}_n$) are defined by

$$E_n(f) = \inf\left\{\|f-g\| : g \in \mathbf{X}_n\right\}.$$

From the Bolzano-Weierstrass theorem (which holds in $\mathbf{X}_n$, since $\dim \mathbf{X}_n < \infty$) it follows easily that the above infimum is a minimum [DL, Theorem 3.1.1]. Thus, the existence of best approximations is ensured:

$$\text{There exist } f_n \in \mathbf{X}_n \text{ such that } E_n(f) = \|f - f_n\|.$$

For $n = 0$ we set $\mathbf{X}_0 = \{0\}$. Of course, this implies that the best approximation f_0 is the zero element and that

$$E_0(f) = \|f\|.$$

An approximation space based on $(\mathbf{X}, \{\mathbf{X}_n\})$ is a set of elements f of $\mathbf{X}$ for which the sequences $\{E_n(f)\}$ belong to a given sequence space. In the general theory of approximation spaces there are almost no restrictions to the choice of the sequence space (see [AL1]). Here we will only consider weighted $\mathbf{l}^q$-spaces, where

$$1 \le q \le \infty \text{ is fixed},$$

since we only need spaces of the type (1.1) and (1.2) in our later applications. We will see that the direct prescription of the weights $\alpha_n > 0$ of the weighted $\mathbf{l}^q$-space is not advantageous in the theory of approximation spaces. Instead of this we define α_n in dependence of another given sequence $\mathcal{A} = \{a_n\}_{n=0}^{\infty}$ which has to satisfy

$$0 = a_0 < 1 = a_1 < a_2 < a_3 < \dots\,, \tag{1.3}$$

$$\lim_{n\to\infty} a_n = \infty\,, \tag{1.4}$$

$$a_{n+1} \le K a_n\,,\ n \in \mathbb{N} = \{1, 2, 3, \dots\}\,, \text{ where } K > 1 \text{ is some constant}. \tag{1.5}$$

Namely, $\alpha_n = a_n(q)$ with the sequence $\mathcal{A}(q) = \{a_n(q)\}_{n=0}^{\infty}$ defined by

$$a_n(q) = \begin{cases} \left(a_{n+1}^q - a_n^q\right)^{1/q} & \text{if } 1 \le q < \infty, \\ a_{n+1} & \text{if } q = \infty. \end{cases} \tag{1.6}$$

In all of what follows we denote by $\mathcal{A}$ a sequence of numbers a_n which satisfy (1.3)–(1.5) and by $\mathcal{A}(q)$ the associated sequence of the numbers $a_n(q)$.

Definition 1.1 *The approximation space $\mathbf{X}_q^{\mathcal{A}} = \mathbf{X}_q^{\mathcal{A}}(\{\mathbf{X}_n\})$ is defined as follows:*

$$\mathbf{X}_q^{\mathcal{A}} = \left\{f \in \mathbf{X} : \{a_n(q)E_n(f)\} \in \mathbf{l}^q\right\}, \quad \textit{endowed with}$$

$$\|f\|_{\mathcal{A},q} = \left\|\{a_n(q)E_n(f)\}_{n=0}^{\infty}\right\|_q,$$

where $\|\,.\,\|_q$ denotes the $\mathbf{l}^q$-norm.

One can easily show that $\mathbf{X}_q^{\mathcal{A}}$ is a normed space which is continuously embedded into $\mathbf{X}$: The last assertion is clear (obviously, $\|f\| \le \|f\|_{\mathcal{A},q}$) and for the proof of the norm properties one only has to take into account that

$$E_n(\lambda f) = |\lambda|\, E_n(f) \quad \text{and} \quad E_n(f+g) \le E_n(f) + E_n(g)\,. \tag{1.7}$$

Remark 1.2 *Obviously, $\{E_n(f)\}$ is a decreasing sequence which converges to zero if and only if f belongs to the closure of $\bigcup \mathbf{X}_n$ in $\mathbf{X}$. This implies that $\mathbf{X}_q^{\mathcal{A}}$ contains only elements f from $\operatorname{clos}\bigcup \mathbf{X}_n$, since, for other f, $\|f\|_{\mathcal{A},q} \geq \inf_n E_n(f)\, \|\{a_n(q)\}_{n=0}^{\infty}\|_q$ cannot be finite. (We have supposed that $\lim_{m\to\infty} a_m = \lim_{m\to\infty} \|\{a_n(q)\}_{n=0}^{m-1}\|_q = \infty$.) Consequently, $\mathbf{X}_q^{\mathcal{A}}$ can be viewed as a space of elements f of $\mathbf{X}$ for which $E_n(f)$ converges to zero with a certain prescribed order of convergence, namely the order $\{a_n(q)E_n(f)\} \in \mathbf{l}^q$. In the case $q = \infty$ this means $E_n(f) = O(a_n^{-1})$.*

One may ask for which weights $\alpha_n > 0$ one can find admissible numbers a_n such that $\alpha_n = a_n(q)$. If we take into account that, for any $\{a_n\}$ satisfying (1.3)–(1.5),

$$\alpha_n = a_n(q) \ \text{ for all } \ n \in \mathbb{N}_0 \quad \text{if and only if} \quad \begin{array}{l} a_n = \left\|\{\alpha_m\}_{m=0}^{n-1}\right\|_q \ \text{ for all } \ n \in \mathbb{N} \\ \text{and, in case } q = \infty,\ \alpha_0 < \alpha_1 < \dots, \end{array}$$

then the answer is clear: The numbers α_n have to satisfy $\alpha_n > 0$, $\alpha_0 = 1$, $\left\|\{\alpha_m\}_{m=0}^{n+1}\right\|_q \leq K\, \|\{\alpha_m\}_{m=0}^{n}\|_q$ and, in the case $q = \infty$, $\alpha_n < \alpha_{n+1}$. Then we have $\alpha_n = a_n(q)$ with $a_n = \left\|\{\alpha_m\}_{m=0}^{n-1}\right\|_q$ satisfying (1.3)–(1.5).

The restriction $\alpha_n < \alpha_{n+1}$ in the case $q = \infty$ is natural: At least $\alpha_n \leq \alpha_{n+1}$ can be assumed without loss of generality, since, for any sequence $\{\alpha_n\} \subseteq (0, \infty)$,

$$E_n(f)\alpha_n \leq E_n(f) \max_{0\leq m\leq n} \alpha_m = E_n(f)\alpha_{m(n)} \leq E_{m(n)}(f)\alpha_{m(n)} \quad \text{for all } \ n\,,$$

i.e., $\left\|\{E_n(f)\alpha_n\}_{n=0}^{\infty}\right\|_\infty = \left\|\left\{E_n(f)\, \|\{\alpha_m\}_{m=0}^{n}\|_\infty\right\}_{n=0}^{\infty}\right\|_\infty$. Now, if we assume $\alpha_n \leq \alpha_{n+1}$, then we may also take the strictly increasing sequence $(2-2^{-n})\alpha_n$ instead of α_n: For the corresponding numbers $a_n = 2(1-2^{-n})\alpha_{n-1}$ we obtain, for $q = \infty$,

$$a_n(q) \sim \alpha_n \quad \text{for all } \ n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}\,. \tag{1.8}$$

(Two expressions A and B which depend on variables v are called equivalent for all v, shortly $A \sim B$ for all v, if $c^{-1}|B| \leq |A| \leq c\,|B|$ for all v, where $0 < c \neq c(v)$. By c we denote, in all of what follows, a positive constant. The concrete value of c is non-important for our purpose and may be different at different places. By $c \neq c(v)$ we indicate that c does not depend on v.)

Obviously, (1.8) implies that the space $\mathbf{X}_q^{\mathcal{A}}$ does not change (in the sense of equivalent norms) if $a_n(q)$ is replaced by α_n in its definition. For this reason it is also interesting to know whether, for given numbers a_n, the numbers $a_n(q)$ can be replaced by simpler equivalent expressions α_n. Often, the following consequence of the mean value theorem gives an answer to this question: If we represent the numbers a_n, $n \in \mathbb{N}$, in the form

$$a_n = a(n)\,, \quad \text{where } \ a \in \mathbf{C}([1,\infty)) \cap \bigcap_{k\in\mathbb{N}} \mathbf{C}^1[k, k+1]$$

is a strictly increasing function satisfying $a'(\xi) \sim a'(n+0)$ for $\xi \in (n, n+1)$, $n \in \mathbb{N}$, then

$$a_n(q) \sim a_n \big[(\ln a)'(n+0)\big]^{1/q} \quad \text{for all } \ n \in \mathbb{N}\,. \tag{1.9}$$

Example 1.3 *In the classical theory of approximation spaces (see [P]) only $a_n = n^s$ ($s > 0$ fixed) is considered. In this case (1.9) yields $a_n(q) \sim (n+1)^{s-(1/q)}$ for $n \in \mathbb{N}_0$, i.e., the corresponding approximation space can be defined equivalently by*

$$\mathbf{X}_q^s = \Big\{ f \in \mathbf{X} : \; \|f\|_{s,q} = \big\| \{(n+1)^{s-(1/q)} E_n(f)\}_{n=0}^{\infty} \big\|_q < \infty \Big\} .$$

In the following theorem we give a list of some important properties of approximation spaces. Thereby, we use the following notation:

For a sequence $\{A_m(f)\}_{m=m_0}^{\infty}$ depending on elements f of some set M we write $\lim\limits_{m\to\infty} A_m(f) = 0$ uniformly in $f \in M$ iff $\lim\limits_{m\to\infty} \sup\limits_{f\in M} |A_m(f)| = 0$.

A sequence $\{u_n\}$ of nonnegative numbers is said to be almost decreasing iff $u_n \le c\, u_m$ for all $n \ge m$, where $c \ne c(n,m)$. (In other words: $\{u_n\}$ is almost decreasing if and only if $u_n \sim v_n$, where v_n is decreasing. Indeed, $u_n \sim v_n$ clearly implies $u_n \le c\, u_m$ for $n \ge m$. For the counterdirection take $v_n = \min\limits_{m\le n} u_m$.)

Theorem 1.4 *Let $\mathcal{A} = \{a_n\}$ and $\mathcal{B} = \{b_n\}$ satisfy (1.3)–(1.5). Then, the following assertions hold true:*

(i) *$\mathbf{X}_q^{\mathcal{A}}$ is a Banach space and the embedding $\mathbf{X}_q^{\mathcal{A}} \subseteq \mathbf{X}$ is compact.*

(ii) *If $q < \infty$, then the closure $\operatorname{clos}_{\mathcal{A},q} \bigcup \mathbf{X}_n$ of $\bigcup \mathbf{X}_n$ in $\mathbf{X}_q^{\mathcal{A}}$ is equal to $\mathbf{X}_q^{\mathcal{A}}$. If $q = \infty$, then $\operatorname{clos}_{\mathcal{A},\infty} \bigcup \mathbf{X}_n = \Big\{ f \in \mathbf{X} : \lim\limits_{n\to\infty} a_{n+1} E_n(f) = 0 \Big\}$.*

(iii) *A subset M of $\operatorname{clos}_{\mathcal{A},q} \bigcup \mathbf{X}_n$ (see (ii)) is relatively compact in $\mathbf{X}_q^{\mathcal{A}}$ if and only if M is bounded in $\mathbf{X}$ and $\lim\limits_{m\to\infty} \|\{a_n(q) E_n(f)\}_{n=m}^{\infty}\|_q = 0$ uniformly in $f \in M$.*

(iv) *A sequence of elements f_k of $\operatorname{clos}_{\mathcal{A},q} \bigcup \mathbf{X}_n$ (see (ii)) is convergent in $\mathbf{X}_q^{\mathcal{A}}$ if and only if it is convergent in $\mathbf{X}$ and relatively compact in $\mathbf{X}_q^{\mathcal{A}}$ (see (iii)).*

(v) *If $a_n \ge c\, b_n$ for all n ($0 < c \ne c(n)$), then $\mathbf{X}_q^{\mathcal{A}}$ is continuously embedded into $\mathbf{X}_q^{\mathcal{B}}$. If $\lim\limits_{n\to\infty} b_n^{-1} a_n = \infty$, then this embedding is compact.*

(vi) *If $r \ge q$, then $\mathbf{X}_q^{\mathcal{A}}$ is continuously embedded into $\mathbf{X}_r^{\mathcal{A}}$. If $1 \le r < q$, then $\mathbf{X}_q^{\mathcal{A}}$ is continuously embedded into $\mathbf{X}_r^{\widetilde{\mathcal{A}}}$, where $\widetilde{\mathcal{A}}$ is any fixed sequence of numbers*

$$0 = \widetilde{a}_0 < 1 = \widetilde{a}_1 < \widetilde{a}_2 < \dots \quad \text{with} \;\; \widetilde{a}_n \sim a_n \ln^{-s} a_{n+1} \;\, (n \in \mathbb{N}) \;\; \text{and} \;\; s > \frac{1}{r} - \frac{1}{q} .$$

(Remark that $\widetilde{a}_n$ satisfies (1.5) with some $\widetilde{K}$ and that $a_n \ln^{-s} a_{n+1} \sim g(a_{n+1})$, where $g(x) = x \ln^{-s} x$ is strictly increasing for sufficiently large $x \ge x_0$.)

(vii) *Let $\mathbf{W}$ be a normed space which is continuously embedded into $\mathbf{X}$ and which contains $\bigcup \mathbf{X}_n$. Suppose that, for some fixed $r \in (0,\infty)$, the so-called Jackson and Bernstein inequalities of order r hold true, i.e., that*

(J) $E_n(f) \le c\, n^{-r} \|f\|_{\mathbf{W}}$ *for all* $f \in \mathbf{W}$ *and all* $n \in \mathbb{N}$, *where* $c \ne c(n,f)$,

(B) $\|f_n\|_{\mathbf{W}} \le c\, n^r \|f_n\|$ *for all* $f_n \in \mathbf{X}_n$ *and all* $n \in \mathbb{N}$, *where* $c \ne c(n,f_n)$.

If $a_n = n^s d_n$ *with some* $0 < s < r$ *and some almost decreasing sequence* $\{d_n\}$*, then* $\mathbf{X}_q^{\mathcal{A}}$ *can be defined equivalently with* $K(f, n^{-r})$ *instead of* $E_n(f)$*, where*

$$K(f,t) = \inf_{g \in \mathbf{W}} \left(\|f - g\| + t\, \|g\|_{\mathbf{W}} \right) \quad (\textit{particularly, } K(f,\infty) = \|f\|)\,.$$

(By "equivalently" we mean $\|f\|_{\mathcal{A},q} \sim \left\| \{a_n(q) K(f, n^{-r})\}_{n=0}^{\infty} \right\|_q$ *for all* $f \in \mathbf{X}$*.)*

(viii) Let $\mathbf{Y}$ *be a Banach space which contains* $\bigcup \mathbf{X}_n$*.*

(a) Suppose that $\mathbf{X}$ *is continuously embedded into* $\mathbf{Y}$*. Then* $\mathbf{X}_q^{\mathcal{A}}$ *is continuously embedded into* $\mathbf{Y}_q^{\mathcal{A}}(\{\mathbf{X}_n\})$*.*

(b) Suppose that $\mathbf{Y}$ *is continuously embedded into* $\mathbf{X}$ *and that*

$$\|f_n\|_{\mathbf{Y}} \le c\, a_n \|f_n\| \quad \textit{for all } f_n \in \mathbf{X}_n \textit{ and all } n \in \mathbb{N}\,,$$

where $c \neq c(n, f_n)$*. If, for some* $\varepsilon > 0$*,* $\{a_n^{\varepsilon} b_n^{-1}\}_{n=1}^{\infty}$ *is almost decreasing, then* $\mathbf{X}_q^{\mathcal{AB}}$ *(*$\mathcal{AB} = \{a_n b_n\}$*) is continuously embedded into* $\mathbf{Y}_q^{\mathcal{B}}(\{\mathbf{X}_n\})$*.*

Remark 1.5 *The assertions (i)–(iv) of Theorem 1.4 are also true if (1.5) is not fulfilled. However, in applications this condition is usually satisfied. In some sense, (1.5) means that the increase of* a_n *is not allowed to be faster than exponential: Exponential increase* $a_n = K^{n-1}$ *(*$n \in \mathbb{N}$*) is still admissible, but faster increase is impossible, since (1.5) implies* $a_n \le K a_{n-1} \le K^2 a_{n-2} \le \ldots \le K^{n-1} a_1 = K^{n-1}$ *(*$n \in \mathbb{N}$*).*

1.2 Examples of approximation spaces

Now we present those two examples of approximation spaces which are of main importance in our later applications. In both examples the underlying space $\mathbf{X}$ is a so-called weighted space of continuous functions,

$$\mathbf{C}_u = \{ f : \mathrm{supp}_* u \to \mathbb{C} \ \text{ such that } fu \in \mathbf{C} := \mathbf{C}[-1,1] \}\,,$$

where $u : [-1,1] \to \mathbb{R}$ is a given continuous function (the weight) for which

$$\mathrm{supp}_* u = \{ x \in [-1,1] : \ u(x) \neq 0 \}$$

is dense in $[-1,1]$. By $fu \in \mathbf{C}$ we mean that fu possesses a continuous extension on $[-1,1]$ (which is also denoted by fu). This implies that the elements f of $\mathbf{C}_u$ are continuous on $\mathrm{supp}_* u$ and that f may have singularities in the zeros of u. It is clear that $\mathbf{C}_u$, endowed with the norm

$$\|f\|_u = \|fu\|\,, \quad \text{where} \quad \|g\| = \sup\{ |g(x)| : x \in [-1,1] \}\,,$$

is a Banach space which is isometrically isomorphic to $\mathbf{C}$.

We consider approximation by algebraic polynomials, i.e., $\mathbf{X}_n$ is equal to

$$\Pi_n = \mathrm{span}\left\{ x^k : \ k = 0, 1, \ldots, n-1 \right\} \qquad (\Pi_0 = \{0\})\,.$$

The corresponding best approximation errors of $f \in \mathbf{C}_u$ are denoted by

$$E_n^u(f) = \inf\left\{\|f - P_n\|_u : P_n \in \Pi_n\right\}.$$

(In particular, $E_0^u(f) = \|f\|_u$.) If $u \equiv 1$, then we write $E_n(f)$ instead of $E_n^u(f)$.

We remark that, in general, the set Π of all algebraic polynomials is not dense in $\mathbf{C}_u$. More precisely:

$$\begin{aligned}&\text{If } f \text{ belongs to the closure } \operatorname{clos}_u \Pi \text{ of } \Pi \text{ in } \mathbf{C}_u \ \bigl(\text{i.e., } \lim_{n\to\infty} E_n^u(f) = 0\bigr),\\ &\text{then } fu \text{ vanishes in all zeros } x_0 \in [-1,1] \text{ of } u.\end{aligned} \tag{1.10}$$

Indeed, $P_k \to f$ in $\mathbf{C}_u$ ($P_k \in \Pi$) means that uP_k converges uniformly on $[-1,1]$ to uf. If $x_0 \in [-1,1]$ is a zero of u, then $(uP_k)(x_0) = 0$ for all k and this implies $(fu)(x_0) = 0$.

For many weights u, the elements f of $\operatorname{clos}_u \Pi$ are just characterized by the property $(fu)(x_0) = 0$, $x_0 \in [-1,1] \setminus \operatorname{supp}_* u$:

Proposition 1.6 *Let $u(x) = B(x) \prod\limits_{i=1}^{N} |x - x_i|^{\alpha_i}$, where $x_i \in [-1,1]$, $\alpha_i > 0$, $N \in \mathbb{N}_0$ (for $N = 0$ we set $u = B$) and $B : [-1,1] \to \mathbb{R}$ is a piecewise continuous function with possible jumps only in the points x_i and with $|B(x)| \geq c > 0$ for all $x \in [-1,1]$. Then,*

$$\operatorname{clos}_u \Pi = \left\{f \in \mathbf{C}_u : (fu)(x_i) = 0 \ \textit{for all} \ i = 1, 2, \ldots, N\right\}.$$

Now we come to the definition of the announced two types of approximation spaces which are of interest in our later applications.

Definition 1.7 *The space $\mathbf{C}_u^0$ is defined by*

$$\mathbf{C}_u^0 = \left\{ f \in \mathbf{C}_u : \|f\|_{u,0} = \sum_{n=0}^{\infty} \frac{E_n^u(f)}{n+1} < \infty \right\}.$$

In the case $u \equiv 1$ we write shortly $\mathbf{C}^0$ and $\|\,.\,\|_0$ instead of $\mathbf{C}_u^0$ and $\|\,.\,\|_{u,0}$, respectively.

Remark 1.8 $\mathbf{C}_u^0 = \mathbf{X}_1^{\mathcal{A}}(\{\Pi_n\})$ *(in the sense of equivalent norms) with*

$$\mathbf{X} = \mathbf{C}_u \quad \textit{and} \quad a_n = \log_2(n+1).$$

This follows from $a_n(1) \sim (n+1)^{-1}$ (see (1.9)). Obviously, (1.3)–(1.5) are satisfied.

Definition 1.9 *Let $0 < \gamma < \infty$ and $\delta \in \mathbb{R}$. The space $\mathbf{C}_u^{\gamma,\delta}$ is defined by*

$$\mathbf{C}_u^{\gamma,\delta} = \left\{ f \in \mathbf{C}_u : \|f\|_{u,\gamma,\delta} = \sup_{n=0,1,\ldots} E_n^u(f)\,(n+1)^{\gamma} \ln^{\delta}(n+2) < \infty \right\}.$$

In the case $u \equiv 1$ we write shortly $\mathbf{C}^{\gamma,\delta}$ and $\|\,.\,\|_{\gamma,\delta}$ instead of $\mathbf{C}_u^{\gamma,\delta}$ and $\|\,.\,\|_{u,\gamma,\delta}$, respectively.

Remark 1.10 $\mathbf{C}_u^{\gamma,\delta} = \mathbf{X}_\infty^{\mathcal{A}}(\{\Pi_n\})$ *(in the sense of equivalent norms) with*

$$\mathbf{X} = \mathbf{C}_u \quad \textit{and} \quad a_n = 2(1-2^{-n}) \max_{1\le m\le n} m^\gamma \log_2^\delta(m+1) \quad (n \in \mathbb{N}).$$

This follows from the fact that, also in the case $\delta < 0$, $n^\gamma \log_2^\delta(n+1)$ is strictly increasing for sufficiently large $n \ge n_0$ (since the derivative of $x^\gamma \log_2^\delta(x+1)$ is positive for $x \ge x_0$) and, consequently,

$$a_n(\infty) = a_{n+1} \sim (n+1)^\gamma \ln^\delta(n+2) \quad \textit{for all } n \in \mathbb{N}_0 . \tag{1.11}$$

It is also clear that the numbers a_n satisfy (1.3)–(1.5).

From Theorem 1.4 we obtain the following properties of $\mathbf{C}_u^0$ and $\mathbf{C}_u^{\gamma,\delta}$.

Theorem 1.11 *The following assertions hold true:*

(i) $\mathbf{C}_u^0$ *and* $\mathbf{C}_u^{\gamma,\delta}$ *are Banach spaces which are compactly embedded into* $\mathbf{C}_u$.

(ii) Π *is dense in* $\mathbf{C}_u^0$.

(iii) The closure of Π *in* $\mathbf{C}_u^{\gamma,\delta}$ *is given by* $\left\{f \in \mathbf{C}_u : \lim_{n\to\infty} E_n^u(f)\, n^\gamma \ln^\delta n = 0\right\}$.

(iv) Let M be a bounded subset of $\mathbf{C}_u$ such that $\lim_{m\to\infty} \sum_{n=m}^{\infty} n^{-1} E_n^u(f) = 0$ uniformly in $f \in M$. Then, M is relatively compact in $\mathbf{C}_u^0$ and a sequence of elements of M is convergent in $\mathbf{C}_u^0$ if and only if it is convergent in $\mathbf{C}_u$.

(v) Let M be a bounded subset of $\mathbf{C}_u$ such that $\lim_{n\to\infty} E_n^u(f)\, n^\gamma \ln^\delta n = 0$ uniformly in $f \in M$. Then, M is relatively compact in $\mathbf{C}_u^{\gamma,\delta}$ and a sequence of elements of M is convergent in $\mathbf{C}_u^{\gamma,\delta}$ if and only if it is convergent in $\mathbf{C}_u$.

(vi) Let $s \ge \gamma$ and, in the case $s = \gamma$, $t > \delta$. Then, the embeddings $\mathbf{C}_u^{s,t} \subseteq \mathbf{C}_u^{\gamma,\delta} \subseteq \mathbf{C}_u^0$ are compact.

(vii) Take u as in Proposition 1.6 and let $v(x) = u(x) \prod_{j=1}^{M} |x-y_j|^{\beta_j}$ $(y_j \in [-1,1])$, where $\beta_j \ge 0$ for all j. Set $\beta = \max_j \beta_j^*$, where $\beta_j^* = \beta_j$ if $y_j \in (-1,1)$ and $\beta_j^* = 2\beta_j$ if $|y_j| = 1$. Then, $\mathbf{C}_v^{\gamma+\beta,\delta}$ is continuously embedded into $\mathbf{C}_u^{\gamma,\delta}$.

We mention that (vii) is a consequence of assertion (viii) of Theorem 1.4, since

$$\|p_n\|_u \le c\, n^\beta \|p_n\|_v \quad \text{for all } p_n \in \Pi_n , \quad \text{where } c \ne c(n, p_n) \tag{1.12}$$

[MT1, Estimate (7.33)]. To be more precise, in (vii) we have to consider $\operatorname{supp}_* v$ as domain of definition of the functions $f \in \mathbf{Y} = \mathbf{C}_u$ to ensure that $\mathbf{Y}$ can be viewed as a linear subspace of $\mathbf{X} = \mathbf{C}_v$.

Let us turn to the question whether the product of functions belonging to approximation spaces of the above type lies again in such an approximation space. The following answer is an immediate consequence of the estimate

$$E_{2n}^{uv}(fg) \le E_{2n-1}^{uv}(fg) \le \|f\|_u \, E_n^v(g) + 2\|g\|_v \, E_n^u(f)\,, \quad f \in \mathbf{C}_u\,,\ g \in \mathbf{C}_v\,,\ n \in \mathbb{N}\,, \tag{1.13}$$

the proof of which is left to the reader. (Approximate fg by $f_n\, g_n$, where $f_n, g_n \in \Pi_n$ are best approximations of f and g, respectively.)

Proposition 1.12 *There is a constant $c \neq c(f,g)$ such that*

$$fg \in \mathbf{C}_{uv}^0 \ \textit{ and } \ \|fg\|_{uv,0} \le c\,\|f\|_{u,0}\,\|g\|_{v,0} \quad \textit{for all} \quad f \in \mathbf{C}_u^0\,,\ g \in \mathbf{C}_v^0\,,$$

$$fg \in \mathbf{C}_{uv}^{\gamma,\delta} \ \textit{ and } \ \|fg\|_{uv,\gamma,\delta} \le c\,\|f\|_{u,\gamma,\delta}\,\|g\|_{v,\gamma,\delta} \quad \textit{for all} \quad f \in \mathbf{C}_u^{\gamma,\delta}\,,\ g \in \mathbf{C}_v^{\gamma,\delta}\,.$$

Of course, the spaces $\mathbf{C}_u^0$ and $\mathbf{C}_u^{\gamma,\delta}$ are only of theoretical interest as long as we do not have practical criteria to check whether a function f belongs to $\mathbf{C}_u^0$ or $\mathbf{C}_u^{\gamma,\delta}$. Such criteria can be given if u has a special form. In the framework of this paper it is sufficient to consider so-called power weights

$$u(x) = \prod_{i=1}^{N} |x - x_i|^{\alpha_i}\,, \quad \text{where } \ x_i \in [-1,1] \ \text{ and } \ \alpha_i > 0 \ \text{ for all } \ i\,. \tag{1.14}$$

For $N = 0$ this means $u \equiv 1$ in agreement with the conventions $\prod_{i\in\emptyset} . = 1$, $\sum_{i\in\emptyset} . = 0$. For such weights u, the space $\mathbf{C}_u^0$ can be characterized with the help of the classical modulus of continuity of $g = fu$. We recall that this modulus is defined by

$$\omega(g,h) = \sup_{x,y\in[-1,1],\,|x-y|\le h} |g(x) - g(y)|\,, \quad h > 0\,. \tag{1.15}$$

Theorem 1.13 *Let u be a weight of the form (1.14). Then, $f \in \mathbf{C}_u$ belongs to $\mathbf{C}_u^0$ if and only if*

$$(fu)(x_i) = 0 \ \text{ for all } i \quad \text{and} \quad \int_0^1 \omega(fu,h)\,\frac{dh}{h} < \infty\,. \tag{1.16}$$

Moreover, $\|f\|_{u,0}^* := \|f\|_u + \int_0^1 \omega(fu,h)\,\frac{dh}{h}$ defines an equivalent norm in $\mathbf{C}_u^0$.

The characterization of the elements of $\mathbf{C}_u^{\gamma,\delta}$ is more difficult and requires the use of assertion (vii) of Theorem 1.4. To define the appropriate space $\mathbf{W}$, we need the notation $\|\,.\,\|_\infty$ for the $\mathbf{L}^\infty(-1,1)$–norm and $\mathbf{AC}_{\mathrm{loc}}^{r-1}(-1,1)$ ($r \in \mathbb{N}$) for the space of all $(r-1)$-times continuously differentiable functions on $(-1,1)$ for which $f^{(r-1)}$ is locally absolutely continuous on $(-1,1)$ (i.e., absolutely continuous on every closed subinterval of $(-1,1)$).

Proposition 1.14 *Let $\varphi(x) = \sqrt{1-x^2}$ and $r \in \mathbb{N}$. If u is a weight of the form (1.14), then the normed space*

$$\mathbf{W} = \mathbf{W}(u,r) = \left\{ f \in \mathbf{AC}_{\mathrm{loc}}^{r-1}(-1,1) :\ \|f\|_{\mathbf{W}} = \|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty < \infty \right\}$$

is continuously embedded into $\mathbf{X} = \mathbf{C}_u$ *(where, in the case* $u(\pm 1) \neq 0$*, the value* $f(\pm 1)$ *of* $f \in \mathbf{W}$ *has to be understood as a limit) and the Bernstein inequality (B) of order* r *(see Theorem 1.4,(vii) with* $\mathbf{X}_n = \Pi_n$*) holds true. If* $\alpha_i \notin \{1, \dots, r\}$ *for all* $x_i \in (-1,1)$*, then also the Jackson inequality (J) of order* r *is true. If there is an* $x_i \in (-1,1)$ *with* $\alpha_i \in \{1, \dots, r\}$*, then (J) holds with* $r - \varepsilon$ *instead of* r*, where* ε *is an arbitrary fixed positive number.*

The following remark shows that the elements of $\mathbf{W}$ are characterized by the property $\|f^{(r)}\varphi^r u\|_\infty < \infty$ (i.e., that $\|fu\|_\infty < \infty$ follows automatically from this property).

Remark 1.15 *Consider* $x_1, \dots, x_N$ *as breakpoints of a partition of* $[-1,1]$ *into subintervals* I_j *and fix inner points* ξ_j *of* I_j*. If* $f \in \mathbf{AC}_{\mathrm{loc}}^{r-1}(-1,1)$ *and* $\|f^{(r)}\varphi^r u\|_\infty < \infty$*, then* $f \in \mathbf{W}(u,r)$ *and*

$$\|f\|_{\mathbf{W}} \sim \|f^{(r)}\varphi^r u\|_\infty + \sum_j \max_{k=0,\dots,r-1} |f^{(k)}(\xi_j)| \,. \tag{1.17}$$

(The constants in this equivalence depend on the choice of the points ξ_j*, but not on* f*.)*

We mention that, for $f \in \mathbf{W}(u,r)$ and $m = 0, \dots, r-1$, Remark 1.15 can be applied to $f^{(m)}$, $\varphi^m u$ and $r-m$ instead of f, u and r. Together with the assertion $\mathbf{W}(\varphi^m u, r-m) \subseteq \mathbf{C}_{\varphi^m u}$ of Proposition 1.14 we obtain

$$f^{(m)} \in \mathbf{C}_{\varphi^m u} \quad \text{and} \quad \left\|f^{(m)}\right\|_{\varphi^m u} \leq c \left(\|f^{(r)}\varphi^r u\|_\infty + \sum_j \max_{k=m,\dots,r-1} |f^{(k)}(\xi_j)| \right)$$

for all $f \in \mathbf{W}(u,r)$ and all $m = 0, \dots, r-1$. Particularly,

$$\|f\|_{\mathbf{W}} \sim \sum_{m=0}^{r} \|f^{(m)}\varphi^m u\|_\infty \quad \text{for all} \quad f \in \mathbf{W}(u,r)\,,$$

since all norms $\|f^{(m)}\varphi^m u\|_\infty$ can be estimated by the right hand side of (1.17) (up to a constant).

The so-called K-functional $K(f,t)$ (see assertion (vii) of Theorem 1.4) which belongs to the spaces $\mathbf{X}$ and $\mathbf{W}$ from Proposition 1.14 is denoted by

$$K_\varphi^r(f,t)_u = \inf_{g \in \mathbf{W}(u,r)} \left(\|f-g\|_u + t\, \|g\|_{\mathbf{W}(u,r)} \right).$$

The behavior of this K-functional for $t \downarrow 0$ is closely connected with the smoothness properties of f. For a precise formulation of this fact we have to define the so-called φ–modulus of smoothness

$$\begin{aligned}
\omega_\varphi^r(f,t)_u :=&\\
&\sup_{0<h\leq t} \|u\,\Delta_{h\varphi}^r f\|_{\mathbf{C}\left([-1+4r^2h^2,1-4r^2h^2]\setminus \bigcup_{x_i\in(-1,1)}(x_i-4rh,\,x_i+4rh)\right)}\\
&+ \inf_{P\in\Pi_r} \|(f-P)u\|_{\mathbf{C}([-1,-1+4r^2t^2]\cap[-1,1])} + \inf_{P\in\Pi_r} \|(f-P)u\|_{\mathbf{C}([1-4r^2t^2,1]\cap[-1,1])}\\
&+ \sum_{x_i\in(-1,1)} \inf_{P\in\Pi_r} \|(f-P)u\|_{\mathbf{C}([x_i-4rt,x_i+4rt]\cap[-1,1])}\,.
\end{aligned}$$

(For $h \geq (2r)^{-1}$ we set $[-1+4r^2h^2, 1-4r^2h^2] := \emptyset$ and $\|\,.\,\|_{\mathbf{C}(\emptyset)} := 0$.) Here we denote by $\Delta^r_{h\varphi} f$ the rth central φ–difference of f,

$$(\Delta^r_{h\varphi} f)(x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^k f\Big(x + \Big(\frac{r}{2} - k\Big) h\varphi(x)\Big) \quad \big(\varphi(x) = \sqrt{1-x^2}\,\big)\,. \tag{1.18}$$

(In Subsection 1.4.6 we will show that $(\Delta^r_{h\varphi} f)(x)$ is well-defined for that values of x which are considered in the $\mathbf{C}$–norm of the first addend of the above φ–modulus: For these x, $\big[x-(rh/2)\varphi(x), x+(rh/2)\varphi(x)\big] \subseteq [-1+2r^2h^2, 1-2r^2h^2] \setminus \bigcup_{x_i \in (-1,1)} (x_i - 3rh,\, x_i + 3rh)$.) It is important that we do not use the classical rth difference $\Delta^r_h f$ (replace $\varphi(x)$ by 1 in (1.18) to obtain its definition) for the definition of the modulus of smoothness. This is attributed to the following well known fact: For the equivalent characterization of a certain convergence order like $E_n(f) = O(n^{-\gamma})$ $(\gamma > 0)$ of the best approximation errors one has to consider less smoothness assumptions in the near of ± 1 than inside $(-1+\varepsilon, 1-\varepsilon)$. However, we mention that, clearly, $\Delta^r_h f$ and $\Delta^r_{h\varphi} f$ are closely related, so that the known properties of $\Delta^r_h f$ (see [DL, Section 2.7]) can be used to obtain properties of $\Delta^r_{h\varphi} f$. For example, the equality

$$\Delta^{r+1}_h f = \Delta^1_h \big(\Delta^r_h f\big) \tag{1.19}$$

is often useful. (1.19) means that $\Delta^r_h = (T_h - T_{-h})^r$, where T_h denotes the operator of translation $f(x) \to f(x + (h/2))$. Thus, for the proof of (1.19) one only has to check $\Delta^r_h = (T_h - T_{-h})^r$ with the help of the binomial theorem.

Proposition 1.16 *Let u be a weight of the form (1.14) and let $r \in \mathbb{N}$. If $\alpha_i \notin \{1, \ldots, r\}$ for all $x_i \in (-1,1)$, then*

$$K^r_\varphi(f, t^r)_u \sim \omega^r_\varphi(f,t)_u + t^r \|f\|_u \quad \textit{for all } f \in \mathbf{C}_u \textit{ and all } t \in (0,1]\,. \tag{1.20}$$

Now we combine Propositions 1.14 and 1.16 with assertion (vii) of Theorem 1.4 to obtain the following characterization of the spaces $\mathbf{C}^{\gamma,\delta}_u$. (If $\alpha_i \in \{1, \ldots, r\}$ for some $x_i \in (-1,1)$, then this characterization does not follow immediately and we refer to Subsection 1.4.7 for its proof.) Thereby, we use the equivalence

$$\sup_{n \in \mathbb{N}} n^\gamma K^r_\varphi(f, n^{-r})_u \ln^\delta(n+1) \sim \sup_{t \in (0,1]} \frac{K^r_\varphi(f,t^r)_u}{t^\gamma} \ln^\delta(1+t^{-1})\,, \tag{1.21}$$

which follows from the monotonicity of $K^r_\varphi(f,t^r)_u$ in t and the fact that every $t \in (0,1]$ lies in some interval $\big((n+1)^{-1}, n^{-1}\big]$.

Theorem 1.17 *Let u be a weight of the form (1.14) and let $\gamma > 0$, $\delta \in \mathbb{R}$. Further, fix some natural number $r > \gamma$. Then, $f \in \mathbf{C}_u$ belongs to $\mathbf{C}^{\gamma,\delta}_u$ if and only if $\omega^r_\varphi(f,t)_u \leq c\, t^\gamma \ln^{-\delta}(1+t^{-1})$ for all $t \in (0,1]$ $(c \neq c(t))$. Moreover, the expression*

$$\|f\|_u + \sup_{t \in (0,1]} \frac{\omega^r_\varphi(f,t)_u}{t^\gamma} \ln^\delta(1+t^{-1}) \tag{1.22}$$

defines an equivalent norm in $\mathbf{C}^{\gamma,\delta}_u$.

Often, the assertions of the following (non-trivial) corollary are more useful than the direct application of Theorem 1.17. Thereby, we use the notation $\mathbf{H}^{k+\alpha}(I)$ ($k \in \mathbb{N}_0$, $\alpha \in (0,1]$, I some compact interval) for the space of all $f : I \to \mathbb{C}$ for which $f^{(k)}$ is Hölder continuous with exponent α,

$$\mathbf{H}^{k+\alpha}(I) = \left\{ f \in \mathbf{C}^k(I) : \|f\|_{\mathbf{H}^{k+\alpha}(I)} := \|f\|_{\mathbf{C}(I)} + \sup_{x,y \in I,\, x \neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} < \infty \right\}.$$

Corollary 1.18 *Let u be a weight of the form (1.14) and $\gamma > 0$, $\delta \in \mathbb{R}$, $k \in \mathbb{N}_0$, $\alpha \in (0,1]$.*

(i) If $\alpha_i < 1$ for all $x_i \in (-1,1)$, then $\left\{ f \in \mathbf{C}^k(-1,1) : f^{(k)} \in \mathbf{C}^{\gamma,\delta}_{\varphi^k u} \right\} \subseteq \mathbf{C}^{\gamma+k,\delta}_u$.

(ii) If $k = 0$ or $\alpha_i < 1$ for all $x_i \in (-1,1)$, then every $f \in \mathbf{AC}^k_{\mathrm{loc}}(-1,1)$ for which, with some constants $0 < c \neq c(h)$, $0 < h_0 \neq h_0(h)$, $c_\pm = c_\pm(h) \in \mathbb{C}$, and $c_i = c_i(h) \in \mathbb{C}$,

$$\left\| f^{(k+1)} \varphi^{k+1} u \right\|_{\mathbf{L}^\infty\left([-1+h^2, 1-h^2] \setminus \bigcup_{x_i \in (-1,1)} (x_i - h,\, x_i + h)\right)} \le c\, h^{\alpha - 1}, \quad 0 < h \le h_0 \quad \text{and}$$

$$\begin{aligned} &\left\| (f^{(k)} - c_-) \varphi^k u \right\|_{\mathbf{L}^\infty(-1,-1+h^2)} + \left\| (f^{(k)} - c_+) \varphi^k u \right\|_{\mathbf{L}^\infty(1-h^2,1)} \\ &\qquad + \sum_{x_i \in (-1,1)} \left\| (f^{(k)} - c_i) \varphi^k u \right\|_{\mathbf{L}^\infty(x_i - h,\, x_i + h)} \le c\, h^\alpha, \quad 0 < h \le h_0 \end{aligned} \tag{1.23}$$

belongs to $\mathbf{C}^{k+\alpha,0}_u$. In the case $\alpha = 1$ the condition (1.23) can be omitted if "$k = 0$ or $\alpha_i < 1$ for all $x_i \in (-1,1)$" is replaced by "$\alpha_i \notin \{1, \ldots, r\}$ for all $x_i \in (-1,1)$".

(iii) If $k = 0$ or $\alpha_i < 1$ for all $x_i \in (-1,1)$, then every $f \in \mathbf{C}^k(-1,1)$ for which, with some constants $0 < c \neq c(h)$, $0 < h_0 \neq h_0(h)$, $c_\pm = c_\pm(h) \in \mathbb{C}$, and $c_i = c_i(h) \in \mathbb{C}$,

$$\left| f^{(k)}(x) - f^{(k)}(y) \right| \le c \frac{|x-y|^\alpha}{\min\left\{ (\varphi^{k+\alpha} u)(x),\, (\varphi^{k+\alpha} u)(y) \right\}}, \quad x, y \in (-1,1) \setminus \{x_i\},$$

$$\begin{aligned} &\left\| (f^{(k)} - c_-) \varphi^k u \right\|_{\mathbf{L}^\infty(-1,-1+h^2)} + \left\| (f^{(k)} - c_+) \varphi^k u \right\|_{\mathbf{L}^\infty(1-h^2,1)} \\ &\qquad + \sum_{x_i \in (-1,1)} \left\| (f^{(k)} - c_i) \varphi^k u \right\|_{\mathbf{L}^\infty(x_i - h,\, x_i + h)} \le c\, h^\alpha, \quad 0 < h \le h_0 \end{aligned}$$

belongs to $\mathbf{C}^{k+\alpha,0}_u$. If $k = 0$, then $f \in \mathbf{C}(-1,1)$ can be replaced by $f \in \mathbf{C}((-1,1) \setminus \{x_i\})$.

(iv) If $f \in \mathbf{C}^{\gamma,\delta}$ and $f(x) \neq 0$ for all $x \in [-1,1]$, then $1/f \in \mathbf{C}^{\gamma,\delta}$.

(v) For every $s > 0$ there exist constants $\gamma = \gamma(s,u) > 0$ and $c = c(s,u) > 0$ such that

$$f \in \mathbf{C}^{\gamma,0}_u, \ \|f\|_{u,\gamma,0} \le c \, \|fu\|_{\mathbf{H}^s} \quad \text{for all } f \in \mathbf{C}_u \text{ with } \begin{array}{l} fu \in \mathbf{H}^s\big([-1,1]\big) \text{ and} \\ (fu)(x_i) = 0, \ i = 1, \ldots, N. \end{array}$$

At the end of this section we present some necessary conditions which have to be satisfied if $f \in \mathbf{C}^{\gamma,\delta}_u$.

Proposition 1.19 *Let $f \in \mathbf{C}_u^{\gamma,\delta}$ (with $\gamma > 0$, $\delta \in \mathbb{R}$, and u as in (1.14)). Then, the following assertions hold true.*

(i) $fu \in \mathbf{H}^s([-1,1])$, $\|fu\|_{\mathbf{H}^s} \le c\,\|f\|_{u,\gamma,\delta}$, *and* $(fu)(x_i) = 0$ *for all* i, *where* $s, c > 0$ *are certain constants depending on* γ, δ, *and* u, *but not on* f.

(ii) *If* $\gamma > k \in \mathbb{N}$, *then* $f \in \mathbf{C}^k\big((-1,1) \setminus \{x_i\}\big)$ *and* $f^{(k)} \in \mathbf{C}^{\gamma-k,\delta}_{\varphi^k u}$.

(iii) *If* I *is a closed subinterval of* $(-1,1) \setminus \{x_i\}_{i=1}^N$, *then* $f \in \mathbf{H}^\gamma(I)$ *if* $\gamma \notin \mathbb{N}$ *and* $\delta \ge 0$, $f \in \mathbf{H}^{\gamma-\varepsilon}(I)$ ($\varepsilon \in (0,\gamma)$ *arbitrary) if* $\gamma \in \mathbb{N}$ *or* $\delta < 0$.

(iv) *Let* $\delta \ge 0$ *and let* I *be a closed subinterval of* $[-1,1]$ *which contains exactly one element* x_i *of*

$$\big\{x_0, x_1, \ldots, x_N, x_{N+1}\big\}\,, \quad x_0 := -1\,, \quad x_{N+1} := 1\,.$$

If $|x_i| < 1$, $\gamma > \alpha_i$, *and* $\gamma - \alpha_i \notin \mathbb{N}$, *then* $f \in \mathbf{H}^{\gamma-\alpha_i}(I)$. *If* $x_i \in \{-1,1\}$, $\gamma > 2\alpha_i$ *(set* $\alpha_i = 0$ *if* $u(x_i) \ne 0$*), and* $(\gamma/2) - \alpha_i \in (k,\, k + (1/2))$ *for some* $k \in \mathbb{N}_0$, *then* $f \in \mathbf{H}^{(\gamma/2)-\alpha_i}(I)$.

1.3 Approximation spaces and unbounded operators

Now we come back to the general framework of Section 1.1. In addition to the space $\mathbf{X}$ and its subspaces $\mathbf{X}_n$ we consider a further Banach space $\mathbf{Y}$ and finite-dimensional linear subspaces $\{0\} = \mathbf{Y}_0 \subseteq \mathbf{Y}_1 \subseteq \mathbf{Y}_2 \subseteq \ldots$. The corresponding approximation spaces are denoted by $\mathbf{Y}_q^{\mathcal{A}} = \mathbf{Y}_q^{\mathcal{A}}(\{\mathbf{Y}_n\})$. In the present section we are interested in the mapping properties of an operator

$$A \in \mathbb{L}\left(\bigcup \mathbf{X}_n\,, \mathbf{Y}\right) \quad \text{with} \quad A(\mathbf{X}_n) \subseteq \mathbf{Y}_n \ \text{ for all } \ n\,. \tag{1.24}$$

($\mathbb{L}(\mathbf{X},\mathbf{Y})$ denotes the space of all linear operators from $\mathbf{X}$ into $\mathbf{Y}$. For the space of all bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$ we will use the notation $\mathcal{L}(\mathbf{X},\mathbf{Y})$. If $\mathbf{X} = \mathbf{Y}$, then we write shortly $\mathbb{L}(\mathbf{X})$ and $\mathcal{L}(\mathbf{X})$.) This operator may be unbounded (or even not defined) in the pair $(\mathbf{X},\mathbf{Y})$. But we assume that we have a certain control over the unboundedness:

$$\|A\|_{\mathbf{X}_n \to \mathbf{Y}} \le c\,a_n \ \text{ with given numbers } a_n \text{ satisfying (1.3)–(1.5).} \tag{1.25}$$

Before we consider unbounded operators, we mention that the situation is much easier if the left hand side of (1.25) is uniformly bounded. In this case one can easily prove the following result.

Proposition 1.20 *Let A be a bounded linear operator from $\bigcup \mathbf{X}_n$ (endowed with the norm of $\mathbf{X}$) into $\mathbf{Y}$ and suppose that $A(\mathbf{X}_n) \subseteq \mathbf{Y}_n$ for all n. Then, A possesses a uniquely determined linear and bounded extension onto the closure of $\bigcup \mathbf{X}_n$ in $\mathbf{X}$ and this extension is bounded in every pair $(\mathbf{X}_q^{\mathcal{A}}, \mathbf{Y}_q^{\mathcal{A}})$ ($\mathcal{A} = \{a_n\}$ satisfying (1.3),(1.4)) of approximation spaces based on $(\mathbf{X},\{\mathbf{X}_n\})$ and $(\mathbf{Y},\{\mathbf{Y}_n\})$, respectively.*

Now we come to the case of unbounded operators A.

Theorem 1.21 *Let A satisfy (1.24) and (1.25). Then, the following assertions hold true.*

(i) There exists a uniquely determined extension $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y})$. This extension has the following property:

$$\begin{aligned} \|A(f-f_n)\|_{\mathbf{Y}} \;&\le\; c\,a_n\|f-f_n\| + c\sum_{m=n}^{\infty} a_m(1)E_m(f) \qquad (1.26)\\ &\text{for } f \in \mathbf{X}_1^{\mathcal{A}}\,,\ f_n \in \mathbf{X}_n\,,\ n \in \mathbb{N}_0 \quad \big(c \ne c(n,f,f_n)\big). \end{aligned}$$

(ii) Let $\mathcal{B} = \{b_n\}$ satisfy (1.3)–(1.5) and suppose that, for some $\varepsilon > 0$, $\{a_n^{\varepsilon} b_n^{-1}\}_{n=1}^{\infty}$ is almost decreasing. Then, $\mathbf{X}_q^{\mathcal{AB}}$ ($\mathcal{AB} = \{a_n b_n\}$) is continuously embedded into $\mathbf{X}_1^{\mathcal{A}}$ and $A \in \mathcal{L}(\mathbf{X}_q^{\mathcal{AB}}, \mathbf{Y}_q^{\mathcal{B}})$.

(iii) If $\mathcal{B}$ satisfies the assumptions of (ii), then the following estimate holds true for all $f \in \mathbf{X}_q^{\mathcal{AB}}$ and all sequences $\{f_n\}_{n=1}^{\infty}$ with $f_n \in \mathbf{X}_n$:

$$\big\|\big\{b_n(q)\,\|A(f-f_n)\|_{\mathbf{Y}}\big\}_{n=1}^{\infty}\big\|_q \le c\,\big\|\big\{(a_n b_n)(q)\,\|f-f_n\|\big\}_{n=1}^{\infty}\big\|_q\,, \qquad (1.27)$$

where $c \ne c(f,\{f_n\})$.

Remark 1.22 *Assertion (i) shows that an operator $A \in \mathbb{L}\big(\bigcup \mathbf{X}_n\,, \mathbf{Y}\big)$ possesses an extension $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y})$ if and only if $\|A\|_{\mathbf{X}_n \to \mathbf{Y}} \le c\,a_n$ ($c \ne c(n)$). In this case the extension is uniquely determined. (Here we may take $\mathbf{Y}_n = A(\mathbf{X}_n)$.) Indeed, if $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y})$, then $\|Af_n\|_{\mathbf{Y}} \le c\,\|f_n\|_{\mathcal{A},1} \le c\,a_n\|f_n\|$ for all $f_n \in \mathbf{X}_n$, since $E_m(f_n) = 0$ for $m \ge n$ and $E_m(f_n) \le \|f_n\|$ for $m < n$.*

We have given the estimates (1.26) and (1.27) since, in many applications, one wants to know how fast the images Af_n of approximations $f_n \in \mathbf{X}_n$ of $f \in \mathbf{X}_1^{\mathcal{A}}$ converge to Af. In particular, it would be interesting to know whether the sum on the right hand side of (1.26) can be replaced by $c\,b_n^{-1}\|f\|_{\mathcal{AB},\infty}$ if $f \in \mathbf{X}_\infty^{\mathcal{AB}}$. The answer is contained in Theorem 1.21. Indeed, (1.27) implies $b_n\big\|A\big(f-f_n^{\text{best}}\big)\big\|_{\mathbf{Y}} \le c\,\|f\|_{\mathcal{AB},\infty}$ ($f_n^{\text{best}} \in \mathbf{X}_n$: best approximations). Together with $\big\|A\big(f_n^{\text{best}}-f_n\big)\big\|_{\mathbf{Y}} \le c\,a_n\|f_n^{\text{best}}-f_n\| \le c\,a_n(E_n(f)+\|f-f_n\|) \le c\,a_n\|f-f_n\|$ we obtain the following.

Corollary 1.23 *Let $A \in \mathbb{L}\big(\bigcup \mathbf{X}_n, \mathbf{Y}\big)$ satisfy (1.25) and take $\mathcal{B} = \{b_n\}$ as in assertion (ii) of Theorem 1.21. Then,*

$$\begin{aligned} \|A(f-f_n)\|_{\mathbf{Y}} \;&\le\; c\,a_n\|f-f_n\| + c\,b_n^{-1}\|f\|_{\mathcal{AB},\infty} \qquad (1.28)\\ &\text{for } f \in \mathbf{X}_\infty^{\mathcal{AB}}\,,\ f_n \in \mathbf{X}_n\,,\ n \in \mathbb{N} \quad \big(c \ne c(n,f,f_n)\big)\,. \end{aligned}$$

Later we will see that the assertions of the following example are very useful for the investigation of Cauchy singular integral operators on $[-1,1]$.

Example 1.24 *Take the notation of Section 1.2 and let $\mathbf{B}_v$ ($v \in \mathbf{C}[-1,1]$, $\overline{\operatorname{supp} v} = [-1,1]$) be the Banach space of all functions $f : \operatorname{supp} v \to \mathbb{C}$ for which*

$$\|f\|_v = \sup\big\{|(fv)(x)| : x \in \operatorname{supp} v\big\} < \infty\,.$$

Taking into account that $\mathbf{C}_u^0 = (\mathbf{C}_u)_1^{\mathcal{A}}(\{\Pi_n\})$ *with* $a_n = \log_2(n+1)$*, it follows from Remark 1.22 that an operator* $A \in \mathbb{L}(\Pi, \mathbf{B}_v)$ *possesses an extension* $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_v)$ *if and only if*

$$\|Ap_n\|_v \leq c\,\|p_n\|_u \log_2(n+1) \quad \text{for all } p_n \in \Pi_n \text{ and all } n \in \mathbb{N}\,, \tag{1.29}$$

where $c \neq c(n, p_n)$*. In this case, the extension is uniquely determined. Moreover, Corollary 1.23, applied to*

$$b_n = 2(1-2^{-n}) \max_{1\leq m\leq n} m^\gamma \log_2^{\delta-1}(m+1) \quad (\gamma \in (0,\infty),\ \delta \in \mathbb{R}),$$

shows that every operator $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_v)$ *has the property*

$$\|A(f-p_n)\|_v \leq c\left(\|f-p_n\|_u + \frac{\|f\|_{u,\gamma,\delta}}{n^\gamma \ln^\delta(n+1)}\right)\ln(n+1) \tag{1.30}$$

($n \in \mathbb{N}$*,* $f \in \mathbf{C}_u^{\gamma,\delta}$*,* $p_n \in \Pi_n$*), where* $c \neq c(n, f, p_n)$*. (We mention that, by (1.11),* $(a_n b_n)(\infty) \sim (n+1)^\gamma \ln^\delta(n+2)$ *and* $a_n^\varepsilon b_n^{-1} \sim \big[\max_{1\leq m\leq n} m^\gamma \log_2^{\delta-\varepsilon-1}(m+1)\big]^{-1}$*.) If*

$$A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_v) \quad \text{and} \quad A(\Pi_n) \subseteq \Pi_{kn} \text{ for all } n \in \mathbb{N}$$

($k \in \mathbb{N}$ *some constant), then it is clear that even* $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{C}_v)$ *(since* $A(\Pi) \subseteq \Pi \subseteq \mathbf{C}_v$ *and, in view of Theorem 1.11,* Π *is dense in* $\mathbf{C}_u^0$*) and that (1.30) implies*

$$A \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})\,.$$

(Take $p_n \in \Pi_{[n/k]}$ *such that* $\|f-p_n\|_u = E_{[n/k]}^u(f)$*;* $[n/k]$*: integer part of* n/k*.)*

1.4 Proofs

1.4.1 Proof of Theorem 1.4

We have already seen that $\mathbf{X}_q^{\mathcal{A}}$ is a normed space which is continuously embedded into $\mathbf{X}$ (see the consideration after Definition 1.1). To prove the completeness of $\mathbf{X}_q^{\mathcal{A}}$ we need the following. Thereby we use the notation $\operatorname{clos} M$ for the closure in $\mathbf{X}$ of a set $M \subseteq \mathbf{X}$.

Lemma 1.25 *If* $M \subseteq \mathbf{X}_q^{\mathcal{A}}$ *is bounded by* $B < \infty$ *in* $\mathbf{X}_q^{\mathcal{A}}$ *(i.e.,* $\|f\|_{\mathcal{A},q} \leq B$ *for all* $f \in M$*), then* $\operatorname{clos} M$ *is again bounded by* B *in* $\mathbf{X}_q^{\mathcal{A}}$*.*

Proof. Clearly, (1.7) implies $|E_n(f) - E_n(g)| \leq E_n(f-g) \leq \|f-g\|$. So we obtain the following fact:

$$\text{If } f_m \stackrel{m\to\infty}{\longrightarrow} f \text{ in } \mathbf{X}\,, \text{ then } E_n(f_m) \stackrel{m\to\infty}{\longrightarrow} E_n(f) \text{ for all } n\,. \tag{1.31}$$

Now, if $f \in \operatorname{clos} M$ and $f_m \in M$ with $f_m \to f$ in $\mathbf{X}$, then, for every fixed k, (1.31) can be applied to obtain the limit $m \to \infty$ on the left hand side of

$$\left\|\{a_n(q)E_n(f_m)\}_{n=0}^k\right\|_q \leq B\,.$$

Thus, $\left\|\{a_n(q)E_n(f)\}_{n=0}^{k}\right\|_q \le B$ for all k, i.e., $\|f\|_{\mathcal{A},q} \le B$. ■

Proof of the completeness of $\mathbf{X}_q^{\mathcal{A}}$. Let $\{f_m\}_{m=1}^{\infty}$ be a Cauchy sequence in $\mathbf{X}_q^{\mathcal{A}}$. Since $\mathbf{X}$ is complete, there exists some $f \in \mathbf{X}$ with $\lim_{m\to\infty} \|f_m - f\| = 0$. From Lemma 1.25 it follows $f \in \mathbf{X}_q^{\mathcal{A}}$. Let $\varepsilon > 0$. Then there exists an $m_0 \in \mathbb{N}$ such that

$$\|f_l - f_m\|_{\mathcal{A},q} \le \varepsilon \quad \text{for all } \ l, m \ge m_0\,.$$

Since $M := \{f_l - f_m : l, m \ge m_0\}$ is bounded by ε in $\mathbf{X}_q^{\mathcal{A}}$ and $f - f_m \in \operatorname{clos} M$ for all $m \ge m_0$, Lemma 1.25 yields $\|f - f_m\|_{\mathcal{A},q} \le \varepsilon$ for all $m \ge m_0$ which shows that f_m converges to f in the norm of $\mathbf{X}_q^{\mathcal{A}}$. ■

Although the following result is well known (see, e.g., [T, Section 2.5.1]), we give its proof for the convenience of the reader.

Lemma 1.26 *Let $M \subseteq \operatorname{clos}\bigcup \mathbf{X}_n$. Then, the following assertions are equivalent.*

(i) M is relatively compact in $\mathbf{X}$ (i.e., every sequence of elements of M contains a subsequence which converges in $\mathbf{X}$).

(ii) M is bounded in $\mathbf{X}$ and $\lim_{n\to\infty} E_n(f) = 0$ uniformly in $f \in M$.

Proof. We recall the well-known Hausdorff theorem which asserts that (i) is equivalent to the following statement: For every $\varepsilon > 0$ there exists a finite ε-net M_ε for M (i.e., a finite set M_ε such that, for every $f \in M$, there is some $f_\varepsilon \in M_\varepsilon$ with $\|f - f_\varepsilon\| < \varepsilon$).
(i)$\Rightarrow$(ii): Let $\varepsilon > 0$ and let $\{f_1, \dots, f_m\}$ be a finite ε-net for M. The assumption $M \subseteq \operatorname{clos}\bigcup \mathbf{X}_n$ implies that there exists some n_0 such that $\max_{i=1,\dots,m} E_n(f_i) < \varepsilon$ for all $n \ge n_0$. Now we obtain, for all $f \in M$ and for $i = i(f)$ with $\|f - f_i\| < \varepsilon$,

$$E_n(f) \le E_n(f - f_i) + E_n(f_i) \le \|f - f_i\| + E_n(f_i) < 2\varepsilon\,, \quad n \ge n_0\,.$$

(ii)$\Rightarrow$(i): Let $\varepsilon > 0$ and choose some $n = n(\varepsilon)$ such that $\sup_{f\in M} E_n(f) < \varepsilon/2$. Then,

$$M_n := \mathbf{X}_n \cap \left\{ g \in \mathbf{X} : \operatorname{dist}(g, M) < \frac{\varepsilon}{2} \right\} \qquad \left(\operatorname{dist}(g, M) := \inf_{f\in M} \|g - f\| \right)$$

is an $(\varepsilon/2)$-net for M. M_n is a bounded subset of the finite-dimensional space $\mathbf{X}_n$. Hence, M_n is relatively compact in $\mathbf{X}$ and this ensures the existence of a finite $(\varepsilon/2)$-net M_ε for M_n. Since M_n is an $(\varepsilon/2)$-net for M, we conclude that M_ε is a finite ε-net for M. ■

Proof of the compactness of the embedding $\mathbf{X}_q^{\mathcal{A}} \subseteq \mathbf{X}$. Let M be a bounded subset of $\mathbf{X}_q^{\mathcal{A}}$. Then M is bounded in $\mathbf{X}$ and the estimate

$$E_n(f)\, a_{n+1} = E_n(f) \left\| \{a_m(q)\}_{m=0}^{n} \right\|_q \le \left\| \{E_m(f)\, a_m(q)\}_{m=0}^{n} \right\|_q \le \|f\|_{\mathcal{A},q} \tag{1.32}$$

shows that $\lim_{n\to\infty} E_n(f) = 0$ uniformly in $f \in M$ (and, particularly, $M \subseteq \operatorname{clos}\bigcup \mathbf{X}_n$). By Lemma 1.26, this implies the relative compactness of M in $\mathbf{X}$. ■

To prove the second assertion of Theorem 1.4 we need the following.

Lemma 1.27 *Let $f \in \mathbf{X}_q^{\mathcal{A}}$ and let $f_n \in \mathbf{X}_n$ such that $\|f - f_n\| = E_n(f)$. Then,*

$$\inf_{g\in\mathbf{X}_n} \|f-g\|_{\mathcal{A},q} = \|f-f_n\|_{\mathcal{A},q} = \left\|\{a_m(q)E_{\max\{n,m\}}(f)\}_{m=0}^{\infty}\right\|_q \quad \textit{for all } n \in \mathbb{N}_0\,. \tag{1.33}$$

Proof. Clearly, $E_m(f-f_n) = E_m(f)$ for $m \geq n$. For $m < n$ we have $E_n(f) = E_n(f-f_n) \leq E_m(f-f_n) \leq \|f-f_n\| = E_n(f)$. Thus,

$$E_m(f-f_n) = E_{\max\{m,n\}}(f) \quad \text{for all } m,n \in \mathbb{N}_0 \tag{1.34}$$

and the second equality of (1.33) is proved. Now it is also clear that the first term in (1.33) is not bigger than the last one. Moreover,

$$\|f-g\|_{\mathcal{A},q} \geq \left\|\{a_m(q)E_{\max\{m,n\}}(f-g)\}_{m=0}^{\infty}\right\|_q = \left\|\{a_m(q)E_{\max\{m,n\}}(f)\}_{m=0}^{\infty}\right\|_q$$

for all $g \in \mathbf{X}_n$, which completes the proof. ■

Proof of assertion (ii). Obviously, $f \in \operatorname{clos}_{\mathcal{A},q}\bigcup \mathbf{X}_n$ iff $\inf\limits_{n\in\mathbb{N}_0}\inf\limits_{g\in\mathbf{X}_n}\|f-g\|_{\mathcal{A},q} = 0$. In view of Lemma 1.27, we have to prove the equivalence

$$\inf_{n\in\mathbb{N}_0}\left\|\{a_m(q)E_{\max\{n,m\}}(f)\}_{m=0}^{\infty}\right\|_q = 0 \iff \inf_{n\in\mathbb{N}_0}\left\|\{a_m(q)E_m(f)\}_{m=n}^{\infty}\right\|_q = 0\,.$$

(Note that the last infimum is zero iff $f \in \mathbf{X}_q^{\mathcal{A}}$ and, in case $q = \infty$, $a_{n+1}E_n(f) \to 0$.) Clearly, the first infimum is bigger than or equal to the second one. Thus, we only have to prove the counterdirection of the above equivalence. So, let the second infimum be zero. We remember that (1.32) implies $\lim\limits_{n\to\infty} E_n(f) = 0$. Consequently, for every $\varepsilon > 0$ there exist $M, N \in \mathbb{N}$ such that

$$\left\|\{a_m(q)E_m(f)\}_{m=M}^{\infty}\right\|_q < \frac{\varepsilon}{2} \quad \text{and} \quad E_N(f)\,a_M < \frac{\varepsilon}{2}\,.$$

We conclude

$$\begin{aligned}\left\|\{a_m(q)E_{\max\{m,N\}}(f)\}_{m=0}^{\infty}\right\|_q &\leq E_N(f)\left\|\{a_m(q)\}_{m=0}^{M-1}\right\|_q + \left\|\{a_m(q)E_m(f)\}_{m=M}^{\infty}\right\|_q \\ &= E_N(f)\,a_M + \left\|\{a_m(q)E_m(f)\}_{m=M}^{\infty}\right\|_q < \varepsilon\,,\end{aligned}$$

i.e., the first infimum in the above equivalence is zero. ■

Before we study relatively compactness in $\mathbf{X}_q^{\mathcal{A}}$, we should investigate convergence.

Lemma 1.28 *Let $\{f_k\}_{k=1}^{\infty} \subseteq \operatorname{clos}_{\mathcal{A},q}\bigcup \mathbf{X}_n$. Then, the following assertions are equivalent.*

(i) $\{f_k\}$ is convergent in $\mathbf{X}_q^{\mathcal{A}}$.

(ii) $\{f_k\}$ converges in $\mathbf{X}$ and $\lim\limits_{m\to\infty}\left\|\{a_n(q)E_n(f_k)\}_{n=m}^{\infty}\right\|_q = 0$ uniformly in $k \in \mathbb{N}$.

Proof. (i)⇒(ii): Let f be the $\mathbf{X}_q^{\mathcal{A}}$-limit of $\{f_k\}$ and let $\varepsilon > 0$. There exists some $k_0 = k_0(\varepsilon)$ such that

$$\left\|\{a_n(q)E_n(f-f_k)\}_{n=0}^{\infty}\right\|_q < \frac{\varepsilon}{2} \quad \text{for all } \ k > k_0 \,.$$

Further, there is some $m_0 = m_0(\varepsilon)$ such that

$$\left\|\{a_n(q)E_n(f)\}_{n=m_0}^{\infty}\right\|_q < \frac{\varepsilon}{2} \quad \text{and} \quad \max_{1\le k\le k_0} \left\|\{a_n(q)E_n(f_k)\}_{n=m_0}^{\infty}\right\|_q < \varepsilon \,. \tag{1.35}$$

Indeed, this is obvious for $q < \infty$ and for $q = \infty$ we take into account that $f, f_k \in \operatorname{clos}_{\mathcal{A},\infty} \bigcup \mathbf{X}_n$, which means $\lim\limits_{n\to\infty} a_{n+1}E_n(f) = \lim\limits_{n\to\infty} a_{n+1}E_n(f_k) = 0$ in view of the second assertion of Theorem 1.4. Now, for $k \in \{1,\dots,k_0\}$ the second part of (ii) is contained in (1.35). For $k > k_0$ we may estimate

$$\begin{aligned} \left\|\{a_n(q)E_n(f_k)\}_{n=m_0}^{\infty}\right\|_q &\le \left\|\{a_n(q)E_n(f_k-f)\}_{n=m_0}^{\infty}\right\|_q + \left\|\{a_n(q)E_n(f)\}_{n=m_0}^{\infty}\right\|_q \\ &< \varepsilon \,. \end{aligned}$$

(ii)⇒(i): Let $f_k \to f$ in $\mathbf{X}$ and let

$$\left\|\{a_n(q)E_n(f_k)\}_{n=m_0}^{\infty}\right\|_q \le \varepsilon \quad \text{for all } \ k \,, \tag{1.36}$$

where $m_0 = m_0(\varepsilon)$ does not depend on k. Then one can show in the same way as in the proof of Lemma 1.25 that also f satisfies

$$\left\|\{a_n(q)E_n(f)\}_{n=m_0}^{\infty}\right\|_q \le \varepsilon \,. \tag{1.37}$$

Estimates (1.36) and (1.37) imply

$$\left\|\{a_n(q)E_n(f-f_k)\}_{n=m_0}^{\infty}\right\|_q \le 2\varepsilon \quad \text{for all } \ k \,. \tag{1.38}$$

On the other hand, we can find some $k_0 = k_0(\varepsilon)$ such that

$$\left\|\{a_n(q)E_n(f-f_k)\}_{n=0}^{m_0-1}\right\|_q \le \|f-f_k\| \left\|\{a_n(q)\}_{n=0}^{m_0-1}\right\|_q < \varepsilon \quad \text{for } \ k \ge k_0 \,. \tag{1.39}$$

From (1.38) and (1.39) we conclude $\|f - f_k\|_{\mathcal{A},q} < 3\varepsilon$, $k \ge k_0$, i.e., $f_k \to f$ in $\mathbf{X}_q^{\mathcal{A}}$. ■

Proof of assertions (iii) and (iv). In view of Lemma 1.28, we only have to prove (iii). Let M be bounded in $\mathbf{X}$ and $\lim\limits_{m\to\infty} \left\|\{a_n(q)E_n(f)\}_{n=m}^{\infty}\right\|_q = 0$ uniformly in $f \in M$ (particularly, $M \subseteq \operatorname{clos}_{\mathcal{A},q} \bigcup \mathbf{X}_n$). Obviously, these two assumptions imply that M is a bounded subset of $\mathbf{X}_q^{\mathcal{A}}$. Now it follows from the first assertion of Theorem 1.4 that M is relatively compact in $\mathbf{X}$. Together with the implication "(ii)⇒(i)" of Lemma 1.28 we obtain that M is relatively compact in $\mathbf{X}_q^{\mathcal{A}}$. To prove the counterdirection, let $M \subseteq \operatorname{clos}_{\mathcal{A},q} \bigcup \mathbf{X}_n$ be relatively compact in $\mathbf{X}_q^{\mathcal{A}}$. Then, clearly, M is bounded in $\mathbf{X}$ and it remains to prove that $\lim\limits_{m\to\infty} \left\|\{a_n(q)E_n(f)\}_{n=m}^{\infty}\right\|_q = 0$ uniformly in $f \in M$. Let us assume that this is not true, i.e., that

$$C := \lim_{m\to\infty} \sup_{f\in M} \left\|\{a_n(q)E_n(f)\}_{n=m}^{\infty}\right\|_q = \inf_{m\ge 0} \sup_{f\in M} \left\|\{a_n(q)E_n(f)\}_{n=m}^{\infty}\right\|_q > 0 \,.$$

Then there exists a sequence $\{f_m\}_{m=0}^\infty \subseteq M$ such that

$$\left\|\{a_n(q)E_n(f_m)\}_{n=m}^\infty\right\|_q \geq \frac{C}{2} \quad \text{for all } m \in \mathbb{N}_0 . \tag{1.40}$$

But $\{f_m\}$ possesses a subsequence $\{f_{m_j}\}_{j=0}^\infty$ which converges in $\mathbf{X}_q^{\mathcal{A}}$. In view of Lemma 1.28, this implies

$$\begin{aligned} \lim_{k\to\infty} \left\|\{a_n(q)E_n(f_{m_k})\}_{n=m_k}^\infty\right\|_q &\leq \lim_{k\to\infty} \left\|\{a_n(q)E_n(f_{m_k})\}_{n=k}^\infty\right\|_q \\ &\leq \lim_{k\to\infty} \sup_{j\geq 0} \left\|\{a_n(q)E_n(f_{m_j})\}_{n=k}^\infty\right\|_q = 0 , \end{aligned}$$

which is in contradiction with (1.40). ■

The proofs of assertions (v)-(viii) of Theorem 1.4 (and also the proof of Theorem 1.21) are based on the following lemma, in which the constant K from assumption (1.5) and the following numbers

$$n(j) = n_{\mathcal{A}}(j) := \max\left\{n \in \mathbb{N} : \, a_n \leq K^j\right\} , \quad j = 0, 1, \ldots \tag{1.41}$$

are needed. (Definition (1.41) makes sense, since $a_1 = 1$ and $\lim\limits_{n\to\infty} a_n = \infty$ ensure that the set $\left\{n \in \mathbb{N} : \, a_n \leq K^j\right\}$ is nonempty and contains only finitely many elements.)

Lemma 1.29 *Let $C \geq 1$ be some constant. Then,*

$$\left\|\{a_n(q)E_n\}_{n=1}^\infty\right\|_q \sim \left\|\left\{K^j E_{n(j)}\right\}_{j=0}^\infty\right\|_q \tag{1.42}$$

for all sequences $\{E_n\}_{n=1}^\infty \subseteq [0, \infty)$ with

$$C^{-1}E_{n(j+1)} \leq E_n \leq CE_{n(j)} \quad \text{for all } n \in [n(j), n(j+1)] \;\; (j = 0, 1, \ldots). \tag{1.43}$$

(The constants in the equivalence (1.42) depend on C, K and q, but not on $\{E_n\}$.)

Proof. We have

$$K^{j-1} < a_{n(j)} \leq K^j , \quad j = 0, 1, 2, \ldots . \tag{1.44}$$

Indeed, $a_{n(j)} \leq K^{j-1}$ would imply $a_{n(j)+1} \leq Ka_{n(j)} \leq K^j$ in contradiction to the definition of $n(j)$. Particularly, the numbers $n(j)$ must be different from each other, i.e.,

$$1 = n(0) < n(1) < n(2) < \ldots .$$

First we consider the case $q = \infty$. In view of (1.44), we have to prove the following equivalence:

$$\sup_{n=1,2,\ldots} a_{n+1}E_n \sim \sup_{j=0,1,\ldots} a_{n(j)}E_{n(j)} .$$

We only have to show $\sup a_{n+1}E_n \leq c \sup a_{n(j)}E_{n(j)}$, since the reverse estimate is obvious. For this aim, we remark that every $n \in \mathbb{N}$ lies in some interval $[n(j), n(j+1))$, so that $a_{n+1} \leq a_{n(j+1)} \leq K^{j+1} < K^2 a_{n(j)}$ and $E_n \leq CE_{n(j)}$.

Now we consider the case $q < \infty$. In view of (1.44), we may estimate

$$c\,K^{qj} \le a^q_{n(j)} - a^q_{n(j-2)} = \sum_{n=n(j-2)}^{n(j)-1} a_n(q)^q \quad \text{for all } j \ge 2\,.$$

Consequently,

$$\sum_{j=2}^{\infty} K^{qj} E^q_{n(j)} \le c \sum_{j=2}^{\infty} E^q_{n(j)} \sum_{n=n(j-2)}^{n(j)-1} a_n(q)^q \le c\,C^{2q} \sum_{j=2}^{\infty} \sum_{n=n(j-2)}^{n(j)-1} a_n(q)^q E^q_n\,.$$

This implies $\sum\limits_{j=0}^{\infty} K^{qj} E^q_{n(j)} \le \text{const} \sum\limits_{n=1}^{\infty} a_n(q)^q E^q_n$. On the other hand,

$$\sum_{n=1}^{\infty} a^q_n(q) E^q_n = \sum_{j=0}^{\infty} \sum_{n=n(j)}^{n(j+1)-1} a_n(q)^q E^q_n \le C^q \sum_{j=0}^{\infty} E^q_{n(j)} \sum_{n=n(j)}^{n(j+1)-1} a_n(q)^q$$

and, again by (1.44), the right hand side can be estimated by $c \sum\limits_{j=0}^{\infty} K^{jq} E^q_{n(j)}$. ■

Proof of assertion (v). Let $a_n \ge c\,b_n$. We choose K large enough, so that $a_{n+1} \le K a_n$ as well as $b_{n+1} \le K b_n$ for all $n \in \mathbb{N}$. The estimate $b_{n_{\mathcal{A}}(j)} \le c\,a_{n_{\mathcal{A}}(j)} \le c\,K^j \le K^{j+j_0}$ shows that

$$n_{\mathcal{A}}(j) \le n_{\mathcal{B}}(j + j_0) \quad \text{for all } j \in \mathbb{N}_0\,,$$

where $j_0 \in \mathbb{N}_0$ is some constant. We conclude that, for all $f \in \mathbf{X}$,

$$\|f\| + \left\| \left\{ K^j E_{n_{\mathcal{B}}(j+j_0)}(f) \right\}_{j=0}^{\infty} \right\|_q \le \|f\| + \left\| \left\{ K^j E_{n_{\mathcal{A}}(j)}(f) \right\}_{j=0}^{\infty} \right\|_q\,.$$

In view of (1.42), the left hand side is equivalent to $\|f\|_{\mathcal{B},q}$ (substitute $i = j + j_0$) while the right hand side is equivalent to $\|f\|_{\mathcal{A},q}$. Thus, $\mathbf{X}^{\mathcal{A}}_q$ is continuously embedded into $\mathbf{X}^{\mathcal{B}}_q$.

Now, let $\lim\limits_{n\to\infty} b_n^{-1} a_n = \infty$. One can easily check that the sequence $\mathcal{C} = \{c_n\}$,

$$c_n := \frac{b_n \min\{b_m^{-1} a_m\}_{m=n}^{\infty}}{\min\{b_m^{-1} a_m\}_{m=1}^{\infty}} \qquad (c_0 := 0)\,,$$

satisfies the assumptions (1.3)$-$(1.5). Moreover, $b_n \le c_n \le c\,a_n$, i.e.,

$$\mathbf{X}^{\mathcal{A}}_q \subseteq \mathbf{X}^{\mathcal{C}}_q \subseteq \mathbf{X}^{\mathcal{B}}_q$$

(continuous embeddings), as we have already proved. Now we show that the embedding $\mathbf{X}^{\mathcal{C}}_q \subseteq \mathbf{X}^{\mathcal{B}}_q$ is compact. Let M be a bounded subset of $\mathbf{X}^{\mathcal{C}}_q$, $\|f\|_{\mathcal{C},q} \le B < \infty$ for all $f \in M$. Obviously, the sequence

$$d_n := b_n\,c_n^{-1} \quad (n \in \mathbb{N})$$

is decreasing and its limit is zero. Consequently, $b_n(q) = (c_n d_n)(q) \le c_n(q)\,d_n$ and

$$\sup_{f\in M} \left\| \{b_n(q) E_n(f)\}_{n=m}^{\infty} \right\|_q \le d_m \sup_{f\in M} \left\| \{c_n(q) E_n(f)\}_{n=m}^{\infty} \right\|_q \le B\,d_m$$

converges to zero for $m \to \infty$. (In particular, $M \subseteq \text{clos}_{\mathcal{B},q} \bigcup \mathbf{X}_n$, in view of assertion (ii).) Now, the relatively compactness of M in $\mathbf{X}_q^{\mathcal{B}}$ follows from assertion (iii). ∎

Proof of assertion (vi). For all $f \in \mathbf{X}$, the $\mathbf{X}_q^{\mathcal{A}}$-norm and the $\mathbf{X}_r^{\mathcal{A}}$-norm is equivalent to

$$\|f\| + \left\| \left\{ K^j E_{n(j)}(f) \right\}_{j=0}^{\infty} \right\|_q \quad \text{and} \quad \|f\| + \left\| \left\{ K^j E_{n(j)}(f) \right\}_{j=0}^{\infty} \right\|_r , \tag{1.45}$$

respectively (see (1.42)). If $r \geq q$, then this yields the assertion because of Jensen's inequality $\| \,.\, \|_r \leq \| \,.\, \|_q$. Let $1 \leq r < q$ and $s > (1/r) - (1/q)$. In view of assertion (v), the space $\mathbf{X}_r^{\widetilde{\mathcal{A}}}$ is independent of the choice of $\widetilde{\mathcal{A}}$ (in the sense of equivalent norms). So it remains to prove the embedding for one special choice of $\widetilde{\mathcal{A}}$. Since $a_n \ln^{-s} a_n$ is strictly increasing for sufficiently large n (with limit ∞), we may choose an $\widetilde{\mathcal{A}}$ with

$$\widetilde{a}_n = a_n \ln^{-s} a_n \quad \text{for all } \; n \geq n_0 \, .$$

Now it is easy to see that $\widetilde{a}_n(q) \leq a_n(q) \ln^{-s} a_n \leq c\, a_n(q) \ln^{-s} a_{n+1}$ for $n \geq n_0$. This shows that, for all $f \in \mathbf{X}$,

$$\|f\|_{\widetilde{\mathcal{A}},r} \leq c \left(\|f\| + \left\| \left\{ a_n(q) E_n(f) \ln^{-s} a_{n+1} \right\}_{n=1}^{\infty} \right\|_r \right) .$$

Now we apply (1.42) with $E_n = E_n(f) \ln^{-s} a_{n+1}$. Taking into account that, in view of (1.44), $\ln^{-s} a_{n(j)+1} \sim (j+1)^{-s}$, we obtain

$$\|f\|_{\widetilde{\mathcal{A}},r} \leq c \left(\|f\| + \left\| \left\{ K^j (j+1)^{-s} E_{n(j)}(f) \right\}_{j=0}^{\infty} \right\|_r \right) .$$

By Hölder's inequality, the rth power of the second addend can be estimated by

$$\left\| \left\{ \left[K^j E_{n(j)}(f) \right]^r \right\}_{j=0}^{\infty} \right\|_{q/r} \left\| \left\{ (j+1)^{-sr} \right\}_{j=0}^{\infty} \right\|_{q/(q-r)} .$$

The rth root of the first factor is, up to $\|f\|$, just the first term in (1.45). The second factor is a real constant, since we have supposed $srq/(q-r) > 1$. ∎

To prove assertion (vii), we need the following three lemmas. Although their proofs are based on arguments which are standard in approximation theory (see, e.g., [DL]), we give it for the convenience of the reader.

Lemma 1.30 (Jackson's theorem) *If (J) holds true, then*

$$E_n(f) \leq c\, K(f, n^{-r}) \quad \textit{for all } \; f \in \mathbf{X} \; \textit{ and all } \; n \in \mathbb{N} \, ,$$

where $c \neq c(f,n)$.

Proof. In view of (1.7) and (J), we have

$$E_n(f) \leq E_n(f-g) + E_n(g) \leq c \left(\|f-g\| + n^{-r} \|g\|_{\mathbf{W}} \right) \quad \text{for all } \; g \in \mathbf{W} \, .$$

Taking the infimum over all g, we obtain the assertion. ∎

Lemma 1.31 (Inverse theorem) *Let $0 = n(-1) < 1 = n(0) \le n(1) \le n(2) \le \dots$ be integers. If (B) holds true, then*

$$K(f, n(j)^{-r}) \le c\, n(j)^{-r} \sum_{i=0}^{j} n(i)^r E_{n(i-1)}(f) \quad \textit{for all } f \in \mathbf{X} \textit{ and all } j \in \mathbb{N}_0\,,$$

where $c \ne c(f, j, \{n(j)\})$.

Proof. Let f_i be best approximations from $\mathbf{X}_{n(i)}$ to $f \in \mathbf{X}$. (Particularly, $f_{-1} = 0$.) If we write $f_j = \sum_{i=0}^{j}(f_i - f_{i-1})$, then we obtain

$$\begin{aligned} K(f, n(j)^{-r}) &\le \|f - f_j\| + n(j)^{-r} \left\| \sum_{i=0}^{j} (f_i - f_{i-1}) \right\|_{\mathbf{W}} \\ &\le E_{n(j)}(f) + n(j)^{-r} \sum_{i=0}^{j} \|f_i - f_{i-1}\|_{\mathbf{W}}\,. \end{aligned}$$

The addends of the last sum can be estimated with the help of (B),

$$\|f_i - f_{i-1}\|_{\mathbf{W}} \le c\, n(i)^r \|f_i - f_{i-1}\| \le c\, n(i)^r \left(E_{n(i)}(f) + E_{n(i-1)}(f)\right) \le c\, n(i)^r E_{n(i-1)}(f).$$

Furthermore, $E_{n(j)}(f) \le n(j)^{-r} n(j)^r E_{n(j-1)}(f) \le n(j)^{-r} \sum_{i=0}^{j} n(i)^r E_{n(i-1)}(f)$. ∎

Lemma 1.32 (Hardy's inequality) *Let $\gamma > 0$ and $K > 1$ be constants and let $\{u_j\}_{j=0}^{\infty}$ be an almost decreasing sequence of nonnegative real numbers. Then,*

$$\left\| \left\{ K^{-\gamma j} u_j \sum_{i=0}^{j} \alpha_i \right\}_{j=0}^{\infty} \right\|_q \le c \left\| \left\{ K^{-\gamma j} u_j\, \alpha_j \right\}_{j=0}^{\infty} \right\|_q \quad \textit{for all } \{\alpha_j\}_{j=0}^{\infty} \subseteq [0, \infty)\,,$$

where $c \ne c(\{\alpha_j\})$.

Proof. First we mention that $u_j \sum_{i=0}^{j} \alpha_i \le c \sum_{i=0}^{j} u_i\, \alpha_i$, so that we may assume $u_j = 1$ for all j. (Otherwise we consider $u_j\, \alpha_j$ instead of α_j.) To prove the assertion in case $u_j = 1$, we define $p \in [1, \infty]$ by $(1/p) + (1/q) = 1$ and choose some $\beta \in (0, \gamma)$. Then, Hölder's inequality gives, for each $j \in \mathbb{N}_0$,

$$\sum_{i=0}^{j} \alpha_i \le \left\| \left\{ K^{\beta i} \right\}_{i=0}^{j} \right\|_p \left\| \left\{ K^{-\beta i} \alpha_i \right\}_{i=0}^{j} \right\|_q \le c\, K^{\beta j} \left\| \left\{ K^{-\beta i} \alpha_i \right\}_{i=0}^{j} \right\|_q\,.$$

Consequently,

$$\begin{aligned} \left\| \left\{ K^{-\gamma j} \sum_{i=0}^{j} \alpha_i \right\}_{j=0}^{\infty} \right\|_q &\le c \left\| \left\{ K^{(\beta-\gamma) j} \left\| \{K^{-\beta i} \alpha_i\}_{i=0}^{j} \right\|_q \right\}_{j=0}^{\infty} \right\|_q \\ &= c \left\| \left\{ K^{-\beta i} \alpha_i \left\| \{K^{(\beta-\gamma) j}\}_{j=i}^{\infty} \right\|_q \right\}_{i=0}^{\infty} \right\|_q \le c \left\| \left\{ K^{-\gamma i} \alpha_i \right\}_{i=0}^{\infty} \right\|_q\,. \end{aligned}$$

∎

Proof of assertion (vii). From Lemma 1.30 it follows

$$\|f\|_{\mathcal{A},q} \le c \left\| \{a_n(q) K(f, n^{-r})\}_{n=0}^{\infty} \right\|_q$$

for all $f \in \mathbf{X}$. It remains to prove the reverse estimate. Let $n(j)$ be defined in (1.41) and set $n(-1) = 0$. Then, in view of (1.42), we have to show that

$$\left\| \{K^j K(f, n(j)^{-r})\}_{j=0}^{\infty} \right\|_q \le c \left\| \{K^j E_{n(j-1)}(f)\}_{j=0}^{\infty} \right\|_q . \tag{1.46}$$

For this aim, we first apply Lemma 1.31 to the left hand side of (1.46),

$$\left\| \{K^j K(f, n(j)^{-r})\}_{j=0}^{\infty} \right\|_q \le c \left\| \left\{ K^j \, n(j)^{-r} \sum_{i=0}^{j} n(i)^r E_{n(i-1)}(f) \right\}_{j=0}^{\infty} \right\|_q . \tag{1.47}$$

Now we mention that, in view of (1.44) and the assumed (almost-)monotonicity of $d_n = n^{-s} a_n$,

$$u_j := K^{(r/s)\, j} \, n(j)^{-r} \sim \left[a_{n(j)} \, n(j)^{-s}\right]^{r/s} = d_{n(j)}^{r/s}$$

is almost decreasing. Writing $K^j \, n(j)^{-r} = K^{-[(r/s)-1]\, j} \, u_j$, we see that Lemma 1.32 can be applied to estimate the right hand side of (1.47). This gives (1.46). ■

Proof of assertion (viii). Part (a) is obvious. (Use that, up to a constant, the best approximation errors in the norm of $\mathbf{Y}$ can be estimated by the best approximation errors in the norm of $\mathbf{X}$.) Part (b) follows from Theorem 1.21, which we will prove in the next subsection. Indeed, consider the embedding operator $A = I : \bigcup \mathbf{X}_n \to \mathbf{Y}$. By assertion (i) of Theorem 1.21, this operator possesses a continuous extension $A : \mathbf{X}_1^{\mathcal{A}} \to \mathbf{Y}$. Clearly, this extension is again the embedding operator, since $\bigcup \mathbf{X}_n$ is dense in $\mathbf{X}_1^{\mathcal{A}}$ (see assertion (ii) of Theorem 1.4). Thus, also its restriction onto $\mathbf{X}_q^{\mathcal{AB}}$, which is considered in assertion (ii) of Theorem 1.21, is the embedding operator. ■

1.4.2 Proof of Theorem 1.21

Proof of assertion (i). To prove existence and uniqueness of the extension $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y})$, we only have to show $\|Af\|_{\mathbf{Y}} \le c \|f\|_{\mathcal{A},1}$ for all $f \in \bigcup \mathbf{X}_n$, since $\bigcup \mathbf{X}_n$ is dense in $\mathbf{X}_1^{\mathcal{A}}$ (see assertion (ii) of Theorem 1.4). Let $f \in \bigcup \mathbf{X}_n$ and let $f_n \in \mathbf{X}_n$ be best approximations to f in the norm of $\mathbf{X}$ (particularly, $f_n = f$ for all sufficiently large n). Further, let $n(j)$ be defined as in (1.41). Then we have $Af - Af_1 = \sum_{j=0}^{\infty} A(f_{n(j+1)} - f_{n(j)})$ (finite sum) and, consequently,

$$\begin{aligned} \|Af - Af_1\|_{\mathbf{Y}} &\le \sum_{j=0}^{\infty} \|A(f_{n(j+1)} - f_{n(j)})\|_{\mathbf{Y}} \\ &\le c \sum_{j=0}^{\infty} a_{n(j+1)} \|f_{n(j+1)} - f_{n(j)}\| \le c \sum_{j=0}^{\infty} K^j E_{n(j)}(f) , \end{aligned}$$

where we took (1.44) into account. Together with (1.42) we obtain

$$\|Af\|_{\mathbf{Y}} \le \|Af - Af_1\|_{\mathbf{Y}} + \|Af_1\|_{\mathbf{Y}} \le c\,\|f\|_{\mathcal{A},1}\,.$$

Now, let $f \in \mathbf{X}_1^{\mathcal{A}}$ and let f_n be an arbitrary element of $\mathbf{X}_n$. Then we have

$$\|A(f-f_n)\|_{\mathbf{Y}} \le c\,\|f-f_n\|_{\mathcal{A},1} = c\sum_{m=0}^{\infty}(a_{m+1}-a_m)E_m(f-f_n)\,.$$

If we write $\sum\limits_{m=0}^{\infty}\cdot = \sum\limits_{m=0}^{n-1}\cdot + \sum\limits_{m=n}^{\infty}\cdot$ and take into account that $E_m(f-f_n) = E_m(f)$ for $m \ge n$ and, clearly, $E_m(f-f_n) \le \|f-f_n\|$, then we obtain (1.26). ∎

Proof of $\mathbf{X}_q^{\mathcal{A}\mathcal{B}} \subseteq \mathbf{X}_1^{\mathcal{A}}$. From assertion (vi) of Theorem 1.4 (or assertion (v), if $q=1$) it follows that $\mathbf{X}_q^{\mathcal{A}\mathcal{B}}$ is continuously embedded into $\mathbf{X}_1^{\widetilde{\mathcal{A}}}$, where

$$\widetilde{a}_n \sim \frac{a_n b_n}{(\ln a_{n+1} + \ln b_{n+1})^s} \quad \text{with} \quad s > 1 - \frac{1}{q} \text{ fixed}\,.$$

We have supposed that $\{a_n^{\varepsilon} b_n^{-1}\}_{n=1}^{\infty}$ is almost decreasing. Particularly, $\varepsilon \ln a_n \le c \ln b_n$ and we conclude

$$\widetilde{a}_n \sim a_n \frac{b_n}{\ln^s b_{n+1}} \ge c\, a_n\,.$$

Thus, $\mathbf{X}_1^{\widetilde{\mathcal{A}}}$ is continuously embedded into $\mathbf{X}_1^{\mathcal{A}}$ (see assertion (v) of Theorem 1.4). ∎

Before we complete the proof of assertion (ii), we should first prove assertion (iii). For this aim, we need the following analogue of Lemma 1.32.

Lemma 1.33 (Hardy's inequality) *Let $K > 1$ be constant. Then,*

$$\left\|\left\{K^j \sum_{i=j}^{\infty}\alpha_i\right\}_{j=0}^{\infty}\right\|_q \le c\left\|\{K^j\alpha_j\}_{j=0}^{\infty}\right\|_q \quad \textit{for all} \quad \{\alpha_j\}_{j=0}^{\infty} \subseteq [0,\infty)\,,$$

where $c \ne c(\{\alpha_j\})$.

Proof. Analogously to the proof of Lemma 1.32. ∎

Proof of assertion (iii). Let $f \in \mathbf{X}_q^{\mathcal{A}\mathcal{B}}$ and $f_n \in \mathbf{X}_n$. Then, $f \in \mathbf{X}_1^{\mathcal{A}}$ and (1.26) implies

$$\begin{aligned}&\left\|\{b_n(q)\,\|A(f-f_n)\|_{\mathbf{Y}}\}_{n=1}^{\infty}\right\|_q \\ &\qquad \le c\left\|\{b_n(q)\,a_n\|f-f_n\|\}_{n=1}^{\infty}\right\|_q + c\left\|\left\{b_n(q)\sum_{m=n}^{\infty}a_m(1)E_m(f)\right\}_{n=1}^{\infty}\right\|_q\,. \qquad (1.48)\end{aligned}$$

The sequence $E_n := \sum_{m=n}^{\infty} a_m(1)E_m(f)$ is decreasing. Thus, (1.42) (with $n(j) = n_{\mathcal{B}}(j)$) can be applied to the second addend of (1.48),

$$\begin{aligned}\left\|\left\{b_n(q)\sum_{m=n}^{\infty} a_m(1)E_m(f)\right\}_{n=1}^{\infty}\right\|_q &\sim \left\|\left\{K^j\sum_{m=n(j)}^{\infty} a_m(1)E_m(f)\right\}_{j=0}^{\infty}\right\|_q \\ &= \left\|\left\{K^j\sum_{i=j}^{\infty}\sum_{m=n(i)}^{n(i+1)-1} a_m(1)E_m(f)\right\}_{j=0}^{\infty}\right\|_q .\end{aligned}$$

Now, Lemma 1.33 shows that the last expression can be estimated by

$$\begin{aligned} c\left\|\left\{K^j\sum_{m=n(j)}^{n(j+1)-1} a_m(1)E_m(f)\right\}_{j=0}^{\infty}\right\|_q &\le c\left\|\left\{K^jE_{n(j)}(f)\sum_{m=n(j)}^{n(j+1)-1} a_m(1)\right\}_{j=0}^{\infty}\right\|_q \\ &\le c\left\|\left\{K^jE_{n(j)}(f)\,a_{n(j+1)}\right\}_{j=0}^{\infty}\right\|_q . \end{aligned} \tag{1.49}$$

The assumption on $\{a_n^{\varepsilon}b_n^{-1}\}$ and the equivalence $b_{n(j)} \sim b_{n(j+1)}$ (see (1.44)) imply $a_{n(j+1)} \le c\,a_{n(j)}$. Indeed,

$$a^{\varepsilon}_{n(j+1)} = \left[a^{\varepsilon}_{n(j+1)}b^{-1}_{n(j+1)}\right] b_{n(j+1)} \le c\left[a^{\varepsilon}_{n(j)}b^{-1}_{n(j)}\right] b_{n(j)} = c\,a^{\varepsilon}_{n(j)} .$$

Thus, in (1.49) we may replace $a_{n(j+1)}$ by $a_{n(j)}$ and for the resulting expression we can apply (1.42) (since $E_n := a_nE_n(f)$ satisfies (1.43)). We obtain

$$\left\|\left\{b_n(q)\sum_{m=n}^{\infty} a_m(1)E_m(f)\right\}_{n=1}^{\infty}\right\|_q \le c\left\|\left\{b_n(q)\,a_nE_n(f)\right\}_{n=1}^{\infty}\right\|_q .$$

Obviously, the right hand side can be estimated by the first addend of (1.48). Finally, it is easy to show that $b_n(q)\,a_n \le (a_nb_n)(q)$, which completes the proof. ∎

Proof of assertion (ii). We have already proved that $\mathbf{X}_q^{\mathcal{AB}}$ is continuously embedded into $\mathbf{X}_1^{\mathcal{A}}$. It remains to show that $A \in \mathcal{L}(\mathbf{X}_q^{\mathcal{AB}}, \mathbf{Y}_q^{\mathcal{B}})$. From (1.27), applied to best approximations f_n of $f \in \mathbf{X}_q^{\mathcal{AB}}$, it follows

$$\left\|\left\{b_n(q)\,\|A(f-f_n)\|_{\mathbf{Y}}\right\}_{n=0}^{\infty}\right\|_q \le c\left(\|Af\|_{\mathbf{Y}} + \|f\|_{\mathcal{AB},q}\right).$$

The left hand side is an upper bound for $\|Af\|_{\mathcal{B},q}$ (since $Af_n \in \mathbf{Y}_n$) and the right hand side is equivalent to $\|f\|_{\mathcal{AB},q}$ (since, in view of (i), $\|Af\|_{\mathbf{Y}} \le c\,\|f\|_{\mathcal{A},1} \le c\,\|f\|_{\mathcal{AB},q}$). ∎

1.4.3 Proof of Proposition 1.6

In the proof of the following lemma we need the well known Jackson theorem (see, e.g., [DL, Theorem 7.6.2])

$$E_n(g) \le c\,\omega\big(g, n^{-1}\big)\,, \quad g \in \mathbf{C}\,,\ n \in \mathbb{N} \qquad (c \ne c(n,g))\,. \tag{1.50}$$

($\omega(g,h)$ is defined in (1.15).)

Lemma 1.34 *Let $B \equiv 1$. Then, there are constants $c > 0$ and $k \in \mathbb{N}$ such that*

$$E^u_{n^k}(f) \le c\left[E_n(fu) + \frac{\|f\|_u}{n}\right] \quad \text{for all } f \in \mathbf{C}_u \text{ with } (fu)(x_i) = 0,\ i = 1, \dots, N$$

($n \in \mathbb{N}$, $c \neq c(n, f)$, $k \neq k(n, f)$).

Proof. Let $N > 0$ (for $N = 0$ we have nothing to prove), let us fix some $\xi = x_j$ and set $\alpha = \alpha_j$. Then we may consider the power weight $v(x) := u(x)/|x - \xi|^\alpha$. The assertion is proved if we have shown that, with some constant k,

$$E^u_{n^k}(f) \le c\left[E^v_n(fu/v) + \frac{\|f\|_u}{n}\right] \quad \text{for } f \in \mathbf{C}_u \text{ with } (fu)(x_i) = 0 \text{ for all } i\,. \tag{1.51}$$

Indeed, we may apply this estimate with n^l instead of n (l large enough) and for the term $E^v_{n^l}(fu/v)$ which now appears on the right hand side we use again the above estimate, but with v instead of u and another x_i. In this way it follows, with $w(x) = v(x)/|x - x_i|^{\alpha_i}$,

$$E^u_{n^{kl}}(f) \le c\left[E^w_n(fu/w) + \frac{\|f\|_u}{n}\right].$$

Repeating this procedure we finally get the assertion. Now we prove (1.51). Set $g(x) = f(x)|x - \xi|^\alpha$ and define

$$\widetilde{P}_n(x) = P_n(x) - P_n(\xi)\,, \quad \text{where} \quad P_n \in \mathbf{\Pi}_n \text{ with } \|g - P_n\|_v = E^v_n(g)\,.$$

Then we have $|P_n(\xi)| = |P_n(\xi) - g(\xi)| = C|(P_n(\xi) - g(\xi))v(\xi)|$ ($C = 1/v(\xi)$) and, consequently, $\|P_n(\xi)\, v\| \le C\|v\|\, E^v_n(g) = c\, E^v_n(g)$. Hence,

$$\left\|g - \widetilde{P}_n\right\|_v \le c\, E^v_n(g) \quad \text{and} \quad \widetilde{P}_n(\xi) = 0\,.$$

Particularly, $Q_n(x) := (x - \xi)^{-1}\widetilde{P}_n(x)$ is a polynomial of degree less than $n - 1$ and, in view of (1.12) (applied with $|\,. - \xi|\, v$ instead of v),

$$\|Q_n\| \le c\, n^\beta \|(\,. - \xi)v\, Q_n\| = c\, n^\beta \left\|\widetilde{P}_n\right\|_v \le c\, n^\beta \|g\|_v = c\, n^\beta \|f\|_u\,,$$

where $\beta > 0$ is some constant (depending only on the exponents α_i). Moreover, we can write

$$\widetilde{P}_n\, v = Q_n\, r\, u \quad \text{with} \quad r(x) = |x - \xi|^{1-\alpha}\, \mathrm{sign}(x - \xi)\,.$$

Let us suppose, for a moment, that $\alpha < 1$. Then r is Hölder continuous with exponent $\mu := 1 - \alpha$. Hence, by (1.50), $E_n(r) \le c\, n^{-\mu}$ and this implies

$$E^u_n(r) \le c\, n^{-\mu}\,. \tag{1.52}$$

Now we choose some natural number l with $l \ge (\beta + 1)/\mu$ and some $R_n \in \Pi_{n^l}$ with $\|r - R_n\|_u = E^u_{n^l}(r)$. Then we obtain

$$\begin{aligned} E^u_{n^{l+1}}(f) &\le E^u_{n^l+n-2}(f) \le \|(f - Q_n R_n)u\| \le \left\|(g - \widetilde{P}_n)v\right\| + \|Q_n(r - R_n)u\| \\ &\le c\, E^v_n(g) + c\, \frac{n^\beta \|f\|_u}{n^{l\mu}} \le c\, E^v_n(g) + c\, \frac{\|f\|_u}{n} \end{aligned}$$

and (1.51) is proved in case $\alpha < 1$. This implies that the lemma is proved if $\alpha_j < 1$ for all j. Now we consider the case $\alpha_j < 2$ for all j. Then it turns out that the proof is the same with one exception: For those $\alpha = \alpha_j$, for which $\alpha \in [1,2)$, the estimate (1.52) has to be proved in a different way. For this aim, we choose some $\eta < 1$ such that $\alpha \in [1, 1+\eta)$. Then the exponent of the weight $\varrho(x) = |x-\xi|^{\alpha-\eta}$ lies in $(0,1)$ and $r(x)\rho(x) = |x-\xi|^{1-\eta}\mathrm{sign}\,(x-\xi)$ is Hölder continuous with exponent $1-\eta$ (particularly, $E_n(r\varrho) \le c\,n^{\eta-1}$ in view of (1.50)) and vanishes in ξ. Thus, we can use what we have already proved:

$$E^{\rho}_{n^k}(r) \le c\left[E_n(r\rho) + \frac{\|r\rho\|}{n}\right] \le \frac{c}{n^{1-\eta}}\,.$$

This implies $E^u_m(r) \le c\,E^\rho_m(r) \le c\,m^{-\mu}$ with $\mu = (1-\eta)/k$. (Use that $m \in [n^k,(n+1)^k)$ for some n.) Similarly one can prove the lemma in the case $\max\alpha_j < 3$, then in the case $\max\alpha_j < 4$, and so on (induction). ■

Proof of Proposition 1.6. We have already seen that $f \in \mathrm{clos}_u\,\Pi$ implies $(fu)(x_i) = 0$, $i = 1,\dots,N$ (see (1.10)). To prove the counterdirection, let $f \in \mathbf{C}_u$ with $(fu)(x_i) = 0$ for all i, and set

$$v(x) = \prod_{i=1}^N |x-x_i|^{\alpha_i}\,.$$

Then it is clear that $f \in \mathbf{C}_v$ and $(fv)(x_i) = 0$ for all i. Now, Lemma 1.34 shows that $\lim\limits_{n\to\infty} E^v_n(f) = \inf\limits_{n\ge 0} E^v_n(f) = 0$ (since $E_n(fv) \to 0$ by (1.50)). But

$$E^v_n(f) \sim E^u_n(f) \quad \text{for all } \; n\,,$$

since $\|f-P_n\|_v \sim \|f-P_n\|_u$ for all n and all $P_n \in \Pi_n$. ■

1.4.4 Proof of Theorem 1.13

Lemma 1.35 *There are constants $c > 0$ and $k \in \mathbb{N}$ ($c \ne c(n,f)$, $k \ne k(n,f)$) such that*

$$E_{n^k}(fu) \le c\left[E^u_n(f) + \frac{\|f\|_u}{n}\right] \quad \text{for all } \; f \in \mathbf{C}_u \; \text{ and all } \; n \in \mathbb{N}\,.$$

Proof. Use that the Hölder continuity of u yields $E_m(u) \le c\,m^{-\mu}$ (by (1.50)) and that (1.12) (applied to 1 and u instead of u and v) implies

$$E_{n+n^l-1}(fu) \le \|fu - f_n u_n\| \le \|(f-f_n)u\| + c\,n^\beta \|f_n\|_u \|u-u_n\|$$

for all $f_n \in \Pi_n$ and all $u_n \in \Pi_{n^l}$. ■

In the following proof we need the well known Markov inequality

$$\|P_n'\| \le (n-1)^2 \|P_n\|\,, \quad P_n \in \Pi_n\,, \; n \in \mathbb{N} \tag{1.53}$$

(see, e.g., [DL, Theorem 4.1.4]).

Proof of Theorem 1.13. The proof of the norm properties of $\| \, . \, \|_{u,0}^*$ is left to the reader. We need the equivalence

$$\sum_{n=1}^{\infty} \frac{E(n)}{n} \sim \sum_{j=0}^{\infty} E(2^j) \quad \text{for all decreasing } E:[1,\infty) \to [0,\infty)\,, \tag{1.54}$$

which follows from (1.42), applied to $q = 1$, $a_n = n$, $K = 2$ and $E_n = n^{-1}E(n)$. This equivalence implies

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{E(n)}{n} &\sim E(1) + \sum_{j=1}^{\infty} E(2^j) \int_{2^{-j}}^{2^{-j+1}} \frac{dh}{h} \\ &\le E(1) + \sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} E(h^{-1}) \frac{dh}{h} = E(1) + \int_0^1 E(h^{-1}) \frac{dh}{h} \quad \text{and} \\ \sum_{n=1}^{\infty} \frac{E(n)}{n} &\sim \sum_{j=0}^{\infty} E(2^j) \int_{2^{-j-1}}^{2^{-j}} \frac{dh}{h} \ge \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} E(h^{-1}) \frac{dh}{h} = \int_0^1 E(h^{-1}) \frac{dh}{h}\,. \end{aligned}$$

The substitution $h = t^\theta$ shows that the last integral can be replaced by $\int_0^1 E(t^{-\theta}) \frac{dt}{t}$, where θ is an arbitrary fixed positive number. So we obtain

$$\sum_{n=1}^{\infty} \frac{E(n)}{n} \sim E(1) + \int_0^1 E(t^{-\theta}) \frac{dt}{t} \quad \text{for all decreasing } E:[1,\infty) \to [0,\infty). \tag{1.55}$$

Now, let k be an arbitrary fixed natural number and let $f \in \mathbf{C}_u$. If we set $\theta = 1/k$ and $E(x) = E^u_{[x^k]}(f)$ ($[x^k]$: integer part of x^k), then we get

$$\sum_{n=1}^{\infty} \frac{E^u_{n^k}(f)}{n} \sim E^u_1(f) + \int_0^1 E^u_{[t^{-1}]}(f) \frac{dt}{t} \quad \text{for all } f \in \mathbf{C}_u\,. \tag{1.56}$$

The right hand side does not depend on k. Consequently, the space $\mathbf{C}^0_u$ does not change (in the sense of equivalent norms) if we define its norm with $E^u_{n^k}(f)$ instead of $E^u_n(f)$. Moreover, it is clear that all elements f of $\mathbf{C}^0_u$ belong to $\mathrm{clos}_u\, \mathbf{\Pi}$, i.e., satisfy $(fu)(x_i) = 0$ for all i (see Proposition 1.6). Together with Lemmas 1.34 and 1.35 we obtain

$$f \in \mathbf{C}^0_u \text{ if and only if } fu \in \mathbf{C}^0 \text{ and } (fu)(x_i) = 0 \text{ for all } i\,, \tag{1.57}$$

where the corresponding norms are equivalent. So it remains to consider the space $\mathbf{C}^0$, i.e., to prove the assertion for $u \equiv 1$. For this aim, let $f \in \mathbf{C}$ and $P_n \in \mathbf{\Pi}_n$ such that $E_n(f) = \|f - P_n\|$. From (1.53) it follows, for all $n \in \mathbb{N}$ and all $x, t \in [-1,1]$,

$$\begin{aligned} |f(x) - f(t)| &\le |f(x) - P_n(x)| + |P_n(x) - P_n(t)| + |P_n(t) - f(t)| \\ &\le 2E_n(f) + \|P_n'\|\,|x-t| \le 2E_n(f) + n^2\|P_n\|\,|x-t| \\ &\le 2E_n(f) + 2n^2\|f\|\,|x-t|\,. \end{aligned}$$

For $|x-t| \le 1$ and $n = \left[|x-t|^{-1/4}\right]$ we obtain

$$|f(x) - f(t)| \le 2E_{[|x-t|^{-1/4}]}(f) + 2\|f\|\,|x-t|^{1/2}\,.$$

Consequently, $\omega(f,h) \le 2E_{[h^{-1/4}]}(f) + 2\|f\|\,h^{1/2}$ for all $h \in (0,1]$. Together with (1.55) (applied to $E(x) = E_{[x]}(f)$ and $\theta = 1/4$) we get

$$\int_0^1 \omega(f,h)\,\frac{dh}{h} \le 2\int_0^1 E_{[h^{-1/4}]}(f)\,\frac{dh}{h} + 4\|f\| \le c\,\|f\|_0\,. \tag{1.58}$$

Thus, the integral on the left hand side is finite if f belongs to $\mathbf{C}^0$. The counterdirection follows from (1.55) and Jackson's theorem (1.50):

$$\sum_{n=1}^{\infty} \frac{E_n(f)}{n} \le c\sum_{n=1}^{\infty} \frac{\omega(f,n^{-1})}{n} \sim \omega(f,1) + \int_0^1 \omega(f,h)\,\frac{dh}{h}\,. \tag{1.59}$$

Obviously, (1.58) and (1.59) imply $\|f\|_0 \sim \|f\| + \int_0^1 \omega(f,h)\frac{dh}{h}$. ∎

1.4.5 Proofs of Proposition 1.14 and Remark 1.15

The proof of Proposition 1.14 is very complicated and cannot be given in the framework of this paper. For (B) we refer to [MT1, Estimate (7.29)]. If $\alpha_i \notin \{1,\dots,r\}$ for all $x_i \in (-1,1)$, then the proof of (J) can be found in [MT2] for the case

$$\mathbf{X} = \mathbf{L}_u^\infty = \left\{f : fu \in \mathbf{L}^\infty(-1,1)\right\} \quad (\text{with } \|f\|_{\mathbf{L}_u^\infty} = \|fu\|_\infty\,)\,, \tag{1.60}$$

i.e., in [MT2] it is not stated that $\mathbf{W}$ is a subset of $\mathbf{C}_u$. So we have to mention that $\mathbf{W} \subseteq \mathbf{C}_u$ is a consequence of Jackson's inequality (J) (with $\mathbf{X} = \mathbf{L}_u^\infty$). Indeed, the Jackson inequality implies that every $f \in \mathbf{W}$ belongs to the closure of Π in $\mathbf{L}_u^\infty$. Thus, there are polynomials f_n with $f_n u \to fu$ uniformly on $(-1,1)$. This implies that fu is uniformly continuous on $(-1,1)$ and, hence, can be continuously extended onto $[-1,1]$. Further, we remark that in [MT2] it is stated that (J) is true for all $f : (-1,1) \setminus \{x_i\}_{i=1}^N \to \mathbb{C}$, for which $f^{(r-1)}$ is absolutely continuous on every closed subinterval of $(-1,1) \setminus \{x_i\}_{i=1}^N$. But if one checks the proof in [MT2], then one can see that additional continuity assumptions are needed in that points $x_i \in (-1,1)$ for which $\alpha_i < r$. (See [DMR, Corollary 3.1] for the exact formulation of sufficient assumptions which ensure (J) in the case of a weight u with only one zero in $(-1,1)$.) For this reason we have only admitted $\mathbf{AC}_{\text{loc}}^{r-1}(-1,1)$–functions in the definition of $\mathbf{W}$. (In this way we do not obtain the biggest possible space $\mathbf{W}$, but to obtain an admissible bigger space one "only" has to look for functions which belong to the completion of $\mathbf{W}$.)

It remains to consider (J) when u has at least one zero x_i inside $(-1,1)$ with integer exponent $\alpha_i \in \{1,\dots,r\}$. In this case we add ε to the exponents belonging to theses zeros $(0 < \varepsilon < 1)$ to obtain a new weight $u(\varepsilon)$. Obviously, $\mathbf{W}$ is continuously embedded into $\mathbf{W}(u(\varepsilon),r)$. Moreover, Jackson's inequality holds for $\mathbf{W}(u(\varepsilon),r)$ and $\mathbf{X} = \mathbf{C}_{u(\varepsilon)}$. Thus,

$$\mathbf{W} \subseteq \mathbf{W}(u(\varepsilon),r) \subseteq \mathbf{C}_{u(\varepsilon)}^{r,0} \quad (\text{continuous embeddings})\,.$$

But $\mathbf{C}^{r,0}_{u(\varepsilon)}$ is continuously embedded into $\mathbf{C}^{r-\varepsilon,0}_u$ (see assertion (vii) of Theorem 1.11) and we obtain Jackson's inequality of order $r-\varepsilon$ for $\mathbf{W}$ and $\mathbf{X} = \mathbf{C}_u$.

Proof of Remark 1.15. First we show that

$$\|fu\|_\infty \le c\left(\|f^{(r)}\varphi^r u\|_\infty + \sum_j \max_{k=0,\ldots,r-1} |f^{(k)}(\xi_j)|\right) \tag{1.61}$$

for all $f \in \mathbf{AC}^{r-1}_{\mathrm{loc}}(-1,1)$. (Particularly, $f \in \mathbf{W}$ if $\|f^{(r)}\varphi^r u\|_\infty < \infty$.) This can be proved by induction, so that we only have to consider the case $r = 1$. Let $x \in (-1,1) \cap \mathrm{supp}_* u$, let $I = I_j$ be that interval which contains x, and let $\xi = \xi_j$ be the corresponding fixed inner point of I. If $x \ge \xi$ and $f \in \mathbf{AC}_{\mathrm{loc}}(-1,1)$, then

$$\begin{aligned} |f(x)| &= \left| f(\xi) + \int_\xi^x f'(t)\,dt \right| \le |f(\xi)| + \int_\xi^x |f'(t)|\,dt \\ &\le |f(\xi)| + \|f'\varphi\, u\|_\infty \int_\xi^x \big[\varphi(t)u(t)\big]^{-1} dt\,. \end{aligned}$$

Let $b = x_i$ be the right endpoint of the interval I (we set $x_{N+1} = 1$ and $\alpha_{N+1} = 0$ if $b = 1 > x_N$). Set $\alpha = \alpha_i$ if $b < 1$ and $\alpha = \alpha_i + \frac{1}{2}$ if $b = 1$. Then we have

$$\varphi(t)u(t) \sim (b-t)^\alpha \ge (b-x)^{\alpha_i}(b-t)^{\alpha-\alpha_i} \quad \text{for all } \ t \in (\xi, x)\,.$$

Consequently,

$$|f(x)| \le |f(\xi)| + c\,\|f'\varphi\, u\|_\infty\,(b-x)^{-\alpha_i} \int_\xi^b (b-t)^{\alpha_i - \alpha}\,dt\,.$$

The last integral is a finite constant. After multiplication by $u(x)$ we obtain

$$|f(x)u(x)| \le c\left(|f(\xi)| + \|f'\varphi\, u\|_\infty\right) \quad \text{for } \ x \in [\xi, b)\,.$$

Analogously, one can prove the same estimate for $x \in I$ with $x < \xi$. Hence, (1.61) is proved. Now it is clear that the left hand side of (1.17) can be estimated by a multiple of the right hand side. To prove the reverse estimate, we have to show that, for fixed $\xi = \xi_j$ and fixed $k \in \{1, \ldots, r-1\}$,

$$|f^{(k)}(\xi)| \le c\left(\|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty\right). \tag{1.62}$$

For that aim, choose some constant $h > 0$ such that $\xi - (kh/2)$ and $\xi + (kh/2)$ are inner points of I_j. One can prove by induction that

$$(\Delta_h^k f)(\xi) = h^k f^{(k)}(\eta)\,, \quad \text{where } \ \eta \in \left(\xi - \frac{kh}{2}, \xi + \frac{kh}{2}\right).$$

(Use the property (1.19) of $\Delta_h^k f$.) Consequently,

$$\begin{aligned} f^{(k)}(\xi) &= f^{(k)}(\xi) - h^{-k}(\Delta_h^k f)(\xi) + h^{-k}(\Delta_h^k f)(\xi) \\ &= f^{(k)}(\xi) - f^{(k)}(\eta) + h^{-k}(\Delta_h^k f)(\xi) = \int_\eta^\xi f^{(k+1)}(t)\,dt + h^{-k}(\Delta_h^k f)(\xi)\,. \end{aligned}$$

For $k = r-1$ this implies $|f^{(r-1)}(\xi)| \le c\left(\|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty\right)$. If $k = r-2$, then $f^{(r-2)}(\xi) - f^{(r-2)}(\eta) = (\xi - \eta) f^{(r-1)}(\tau)$. Our considerations for the case $k = r-1$ can be repeated with τ instead of ξ and so we conclude $|f^{(r-2)}(\xi)| \le c\left(\|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty\right)$. Now we can proceed in an analogous fashion to obtain (1.62) for $k = r-3$ and so on. ■

1.4.6 Proof of Proposition 1.16

Lemma 1.36 *Let $h < (2r)^{-1}$. For $x \in [-1+4r^2h^2, 1-4r^2h^2] \setminus \bigcup_{x_i \in (-1,1)} (x_i - 4rh,\, x_i + 4rh)$,*

$$\left[x - \frac{rh}{2}\,\varphi(x),\, x + \frac{rh}{2}\,\varphi(x)\right] \subseteq \left[\,-1 + 2r^2h^2,\, 1 - 2r^2h^2\right] \setminus \bigcup_{x_i \in (-1,1)} (x_i - 3rh,\, x_i + 3rh)\,.$$

Proof. The function $x + (rh/2)\varphi(x)$ is increasing on $[0, 1-4r^2h^2]$ since its derivative is positive on this interval. Thus,

$$\begin{aligned}
\max_{x \in [0,1-4r^2h^2]} \left(x + \frac{rh}{2}\varphi(x)\right) &= 1 - 4r^2h^2 + \frac{rh}{2}\sqrt{1 - (1-4r^2h^2)^2} \\
&= 1 - 4r^2h^2 + \frac{rh}{2}\sqrt{8r^2h^2 - 16r^4h^4} \\
&\le 1 - 4r^2h^2 + \sqrt{2}\, r^2h^2 < 1 - 2r^2h^2\,.
\end{aligned}$$

Further, for $x \in [0, 1-4r^2h^2]$, $x - (rh/2)\varphi(x) \ge -rh/2 > -1/4 > -1 + 2r^2h^2$, and we obtain

$$\left[x - \frac{rh}{2}\,\varphi(x),\, x + \frac{rh}{2}\,\varphi(x)\right] \subseteq \left[\,-1 + 2r^2h^2,\, 1 - 2r^2h^2\right] \quad \text{for all } \ x \in [0,\, 1 - 4r^2h^2]\,.$$

Clearly, the same can be proved for $x \in [-1 + 4r^2h^2,\, 0]$ and it remains to mention that, obviously, $x \notin (x_i - 4rh,\, x_i + 4rh)$ $(i \in \{1, \ldots, N\})$ implies $y \notin (x_i - 3rh,\, x_i + 3rh)$ for all $y \in \left[x - (rh/2)\varphi(x),\, x + (rh/2)\varphi(x)\right]$. ■

Proof of Proposition 1.16. We will conclude the assertion from a known result which is given in [DMR, Theorem 2.3] for the case of a weight u with at most one zero inside $(-1,1)$ (for the sake of simplicity) and which can be proved (following the same lines as in [DMR]) also for weights u as in Proposition 1.16:

$$\begin{aligned}
\inf_{g \in \mathbf{W}(u,r)} \left[\|f-g\|_u + t^r \left\|g^{(r)}\varphi^r u\right\|_\infty\right] &\le c\,\omega_\varphi^r(f,t)_u \\
&\le c\left(\inf_{g \in \mathbf{W}(u,r)} \left[\|f-g\|_u + t^r \left\|g^{(r)}\varphi^r u\right\|_\infty\right] + t^r \|f\|_u\right)
\end{aligned}$$

for all $f \in \mathbf{C}_u$ and all $0 < t \le t_0$, where $t_0 > 0$ is some sufficiently small constant. The last expression in the above estimates is equivalent to $K_\varphi^r(f,t)_u$, since, for all $g \in \mathbf{W}(u,r)$ and all $0 < t \le 1$,

$$\|f-g\|_u + t^r \|g\|_{\mathbf{W}(u,r)} \ \ge\ t^r\left(\|f-g\|_u + \|g\|_u\right) \ge t^r \|f\|_u \quad \text{and} \tag{1.63}$$

$$\begin{aligned}
\|f-g\|_u + t^r \|g\|_{\mathbf{W}(u,r)} &\le \|f-g\|_u + t^r\left(\|g-f\|_u + \|f\|_u\right) + t^r \left\|g^{(r)}\varphi^r u\right\|_\infty \\
&\le 2\left(\|f-g\|_u + t^r \left\|g^{(r)}\varphi^r u\right\|_\infty\right) + t^r \|f\|_u\,.
\end{aligned}$$

Thus, (1.20) holds for $t < t_0$. In view of (1.63),

$$K^r_\varphi(f,t)_u \sim \|f\|_u \quad \text{for} \quad t \in [t_0, 1]\,.$$

The same is true with $K^r_\varphi(f,t)_u$ replaced by $\omega^r_\varphi(f,t)_u + t^r\,\|f\|_u$, since Lemma 1.36 implies

$$\omega^r_\varphi(f,t)_u \le c\,\|f\|_u \quad \text{for all} \;\; f \in \mathbf{C}_u \;\; \text{and all} \;\; t \in (0,1]\,.$$

Here we took into account that

$$1 \pm x \sim 1 \pm y \quad \text{for} \;\; x \in [-1 + 4r^2h^2, 1 - 4r^2h^2] \;\; \text{and} \;\; y \in \left[x - \frac{rh}{2}\,\varphi(x),\, x + \frac{rh}{2}\,\varphi(x)\right]$$

(use $\sqrt{1+x} \ge 2rh \ge rh\sqrt{1-x}$ and, consequently, $1 + x + (rh/2)\varphi(x) \le 3(1+x)/2$, $1 + x - (rh/2)\varphi(x) \ge (1+x)/2$; analogously for $1 - x$) and

$$|x - x_i| \sim |y - x_i| \quad \text{for} \;\; x \notin (x_i - 4rh,\, x_i + 4rh) \;\; \text{and} \;\; y \in \left[x - \frac{rh}{2}\,\varphi(x),\, x + \frac{rh}{2}\,\varphi(x)\right]$$

(since $|y - x| \le (rh)/2 \le |x - x_i|/8$). ∎

1.4.7 Proof of Theorem 1.17

The proof of the norm properties of expression (1.22) is left to the reader. If $\alpha_i \notin \{1, \ldots, r\}$ for all $x_i \in (-1,1)$, then we have already seen that the equivalence of (1.22) and $\|f\|_{u,\gamma,\delta}$ follows from assertion (vii) of Theorem 1.4. If there is an $x_i \in (-1,1)$ with $\alpha_i \in \{1, \ldots, r\}$, then one has to modify the proof, since we have no Jackson inequality of order r and no equivalence (1.20) in this case. We will not present the details (which are rather technical) and simply refer to [DMR] (see also [MT3, Theorem 1.4]) for the proof of the following estimates, which hold true for arbitrary weights of the form (1.14):

$$E^u_n(f) \;\le\; c\,\omega^r_\varphi\big(f, (n-1)^{-1}\big)_u \quad \text{for all} \quad n > r\,, \tag{1.64}$$

$$\omega^r_\varphi\big(f, n^{-1}\big)_u \;\le\; c\,n^{-r} \sum_{k=1}^{n+1} k^{r-1} E^u_k(f) \quad \text{for all} \quad n \in \mathbb{N}\,. \tag{1.65}$$

From (1.64) it follows that, for all $f \in \mathbf{C}_u$,

$$\begin{aligned} \|f\|_{u,\gamma,\delta} &\le c\,\|f\|_u + c \sup_{n>r} (n-1)^\gamma \omega^r_\varphi\big(f, (n-1)^{-1}\big)_u \ln^\delta n \\ &\le c\,\|f\|_u + c \sup_{t\in(0,1]} \frac{\omega^r_\varphi(f,t)_u}{t^\gamma} \ln^\delta(1 + t^{-1})\,. \end{aligned}$$

On the other hand, (1.65) shows that

$$\begin{aligned} \omega^r_\varphi\big(f, n^{-1}\big)_u &\le c\,n^{-r} \|f\|_{u,\gamma,\delta} \sum_{k=1}^{n+1} k^{r-\gamma-1} \ln^{-\delta}(k+1) \\ &\le c\,n^{\varepsilon - r} \|f\|_{u,\gamma,\delta} \ln^{-\delta}(n+1) \sum_{k=1}^{n+1} k^{r-\varepsilon-\gamma-1} \le c\,\frac{\|f\|_{u,\gamma,\delta}}{n^\gamma \ln^\delta(n+1)} \end{aligned}$$

for all $f \in \mathbf{C}_u$ and all $n \in \mathbb{N}$. It remains to mention that (1.21) is also true with $K^r_\varphi(f,t^r)_u$ replaced by $\omega^r_\varphi(f,t)_u$. (Remark that also $\omega^r_\varphi(f,t)_u$ is increasing in t). ∎

1.4.8 Proof of Corollary 1.18

If we would have a Jackson inequality of the type $E_n^u(f) \le c\, n^{-r} \|f^{(r)} \varphi^r u\|_\infty$, $f \in \mathbf{W}(u,r)$, then assertion (i) would follow by a standard argument (see the proof of Lemma 1.37 below). Unfortunately, in general the above inequality is only true with $\|f^{(r)}\varphi^r u\|_\infty$ replaced by $\|f\|_{\mathbf{W}(u,r)}$ if we consider weights u having zeros x_i inside $(-1,1)$ (where $\alpha_i \notin \{1,\dots,r\}$ for these x_i; see Proposition 1.14). However, if $\alpha_i < 1$ for all $x_i \in (-1,1)$, then one can use a nice fact from the proof of Jackson's inequality with $\|f\|_{\mathbf{W}(u,r)}$ given in [MT2], namely the Jackson inequality for a certain modified weight u_n, to show that the inequality is true with $\|f^{(r)}\varphi^r u\|_\infty$ on the right hand side. Even more:

Lemma 1.37 *If $\alpha_i < 1$ for all $x_i \in (-1,1)$, then*

$$E_n^u(f) \le \frac{c}{n^r}\, E_{n-r}^{\varphi^r u}\big(f^{(r)}\big) \quad \textit{for all } \; f \in \mathbf{W}(u,r) \; \textit{ and all } \; n \ge r\,, \tag{1.66}$$

where $c \ne c(n,f)$ and $E_m^{\varphi^r u}(\,.\,)$ denotes the error of best approximation by polynomials of degree $< m$ in the norm of $\mathbf{L}^\infty_{\varphi^r u}$ (see (1.60)).

Proof. We only need to prove (1.66) for $r = 1$, since then the assertion follows by induction. Moreover, it is enough to show

$$E_n^u(f) \le \frac{c}{n}\, \|f'\varphi\, u\|_\infty\,, \quad f \in \mathbf{W}(u,1)\,, \quad n \in \mathbb{N}\,, \tag{1.67}$$

since this can be applied to $E_n^u(f - p_n)$ (which is equal to $E_n^u(f)$), where $p_n \in \Pi_n$ such that $\|(f' - p_n')\varphi\, u\|_\infty = E_{n-1}^{\varphi\, u}\big(f'\big)$. Choose $C \in (0,1)$ such that the intervals $[x_i - 2C,\, x_i + 2C]$ are disjoint, $-1 + C < x_1 - 2C$ if $x_1 > -1$, and $1 - C > x_N + 2C$ if $x_N < 1$. We will first prove that (1.67) holds true with $E_n^u(f)$ replaced by

$$\mathcal{E}_n^u(f) := \inf_{p_n \in \Pi_n} \|(f - p_n)u\|_{\mathbf{C}\left([-1+Cn^{-2},1-Cn^{-2}]\setminus \bigcup\limits_{x_i\in(-1,1)}(x_i - Cn^{-1}, x_i + Cn^{-1})\right)}\,.$$

For this aim we define

$$f_n(x) \;=\; f(x) \quad \text{for } \; x \in [-1 + Cn^{-2}, 1 - Cn^{-2}] \setminus \bigcup_{x_i\in(-1,1)} (x_i - Cn^{-1},\, x_i + Cn^{-1})\,,$$

$$f_n(x) \;=\; \begin{cases} f(-1 + Cn^{-2})\,, & x < -1 + Cn^{-2}\,, \\[1ex] f(1 - Cn^{-2})\,, & x > 1 - Cn^{-2}\,, \\[1ex] \dfrac{f(x_i + Cn^{-1})}{2Cn^{-1}}(x - x_i + Cn^{-1}) \\ \qquad -\dfrac{f(x_i - Cn^{-1})}{2Cn^{-1}}(x - x_i - Cn^{-1})\,, & \begin{array}{l} x \in (x_i - Cn^{-1},\, x_i + Cn^{-1}) \\ \text{for some } x_i \in (-1,1)\,. \end{array} \end{cases}$$

Further, we define the modified weight

$$u_n(x) = \prod_{x_i\in\{-1,1\}} \big(|x - x_i|^{\alpha_i} + n^{-2\alpha_i}\big) \prod_{x_i\in(-1,1)} \big(|x - x_i|^{\alpha_i} + n^{-\alpha_i}\big)\,. \tag{1.68}$$

Applying the following Jackson inequality for u_n,

$$E_n^{u_n}(f) \le \frac{c}{n} \|f'\varphi\, u_n\|_\infty \,, \quad f \in \mathbf{AC}_{\mathrm{loc}}(-1,1)\,, \quad n \in \mathbb{N}$$

(see [MT2, Theorem 1.1] and remark that, for power weights u with positive exponents, the weight u_n defined in [MT2] is equivalent to the right side of (1.68); see [MT1, p.68]) to f_n and taking into account that $\mathcal{E}_n^u(f) = \mathcal{E}_n^u(f_n) \le E_n^{u_n}(f_n)$, we obtain

$$\begin{aligned}
\mathcal{E}_n^u(f) \;&\le\; \frac{c}{n} \left\| f_n' \varphi\, u_n \right\|_\infty \\
&=\; \frac{c}{n} \Bigg[\left\| f'\varphi\, u_n \right\|_{\mathbf{L}^\infty\left([-1+Cn^{-2},1-Cn^{-2}]\setminus \underset{x_i\in(-1,1)}{\cup}(x_i-Cn^{-1},\, x_i+Cn^{-1})\right)} + \\
&\qquad \frac{n}{2C} \max_{x_i\in(-1,1)} \left| f(x_i+Cn^{-1}) - f(x_i-Cn^{-1}) \right| \|\varphi\, u_n\|_{\mathbf{L}^\infty(x_i-Cn^{-1},\, x_i+Cn^{-1})} \Bigg] .
\end{aligned}$$

Together with the equivalences

$$\|\varphi\, u_n\|_{\mathbf{L}^\infty(x_i-Cn^{-1},\, x_i+Cn^{-1})} \sim n^{-\alpha_i} \quad \text{for } \; x_i \in (-1,1)\,,$$

$$u_n(x) \sim u(x) \quad \text{for } \; x \in [-1+Cn^{-2}, 1-Cn^{-2}] \setminus \underset{x_i\in(-1,1)}{\cup} (x_i - Cn^{-1},\, x_i + Cn^{-1})$$

and the estimate

$$\begin{aligned}
\left| f(x_i+Cn^{-1}) - f(x_i-Cn^{-1}) \right| \;&\le\; \int_{x_i-Cn^{-1}}^{x_i+Cn^{-1}} |f'(t)|\, dt \le c\, \|f'\varphi\, u\|_\infty \int_{x_i-Cn^{-1}}^{x_i+Cn^{-1}} \frac{dt}{|t-x_i|^{\alpha_i}} \\
&\le\; c\, n^{\alpha_i - 1} \|f'\varphi\, u\|_\infty
\end{aligned}$$

($x_i \in (-1,1)$) we get

$$\mathcal{E}_n^u(f) \le \frac{c}{n} \|f'\varphi\, u\|_\infty \,, \quad f \in \mathbf{W}(u,1)\,.$$

Let $P_n \in \Pi_n$ such that

$$\|(f-P_n)u\|_{\mathbf{C}\left([-1+Cn^{-2},1-Cn^{-2}]\setminus \underset{x_i\in(-1,1)}{\cup}(x_i-Cn^{-1},x_i+Cn^{-1})\right)} = \mathcal{E}_n^u(f)\,.$$

If we apply the Remez inequality

$$\|p_m u\| \le c\, \|p_m u\|_{\mathbf{C}\left([-1+Lm^{-2},\, 1-Lm^{-2}]\setminus \underset{x_i\in(-1,1)}{\cup}(x_i-Lm^{-1},\, x_i+Lm^{-1})\right)} \,, \quad p_m \in \Pi_{m+1}\,, \tag{1.69}$$

which holds for any fixed $L > 0$ and for all $m \in \mathbb{N}$ for which the measure of the set $[-1+Lm^{-2},\, 1-Lm^{-2}]\setminus \underset{x_i\in(-1,1)}{\cup} (x_i - Lm^{-1},\, x_i + Lm^{-1})$ is positive ([MT1, (6.10)] applied to $p_m(\cos t)\, u(\cos t)$; see also [MT1, Section 7.2]), where we take $m = 2^{j+1}n$ and $L = 4C$, then we obtain

$$\begin{aligned}
&\sum_{j=0}^\infty \left\| P_{2^{j+1}n} - P_{2^j n} \right\|_u \\
&\le c \sum_{j=0}^\infty \left\| \left(P_{2^{j+1}n} - P_{2^j n}\right) u \right\|_{\mathbf{C}\left([-1+C(2^jn)^{-2},1-C(2^jn)^{-2}]\setminus \underset{x_i\in(-1,1)}{\cup}(x_i-2C(2^jn)^{-1},\, x_i+2C(2^jn)^{-1})\right)} \\
&\le c \sum_{j=0}^\infty \left(\mathcal{E}_{2^{j+1}n}^u(f) + \mathcal{E}_{2^j n}^u(f) \right) \le \frac{c}{n} \|f'\varphi\, u\|_\infty
\end{aligned}$$

for all $n \in \mathbb{N}$. Since the series $\sum_{j=0}^{\infty}\left(P_{2^{j+1}n} - P_{2^j n}\right)$ converges uniformly to $f - P_n$ on every closed subinterval of $(-1,1) \setminus \{x_i\}$, we conclude

$$\|f - P_n\|_u \leq \frac{c}{n}\,\|f'\varphi\, u\|_\infty$$

and the lemma is proved. ■

Proof of assertion (i). This assertion follows immediately from (1.66). ■

Proof of assertion (ii). Let $x \in [-1 + 4r^2h^2, 1 - 4r^2h^2] \setminus \bigcup_{x_i \in (-1,1)} (x_i - 4rh,\, x_i + 4rh)$ $(h \leq h_0)$. At the end of the proof of Proposition 1.16 we have seen that

$$(\varphi^{k+1}u)(x) \sim (\varphi^{k+1}u)(y) \quad \text{for all} \quad y \in \left[x - \frac{rh}{2}\,\varphi(x),\, x + \frac{rh}{2}\,\varphi(x)\right]. \tag{1.70}$$

Together with the assumption on $f^{(k+1)}$ we obtain

$$\begin{aligned}\left|\left(\Delta^1_{h\varphi} f^{(k)}\right)(x)\right| &\leq \left(\varphi^{k+1}u\right)^{-1}(x) \int_{x-(h/2)\varphi(x)}^{x+(h/2)\varphi(x)} \left|f^{(k+1)}\varphi^{k+1}u\right|(y)\, dy \\ &\leq c\, h^{\alpha-1}\left(\varphi^{k+1}u\right)^{-1}(x) \int_{x-(h/2)\varphi(x)}^{x+(h/2)\varphi(x)} dy = c\, h^{\alpha}\left(\varphi^{k}u\right)^{-1}(x)\,,\end{aligned}$$

where we took Lemma 1.36 into account. If also (1.23) is satisfied, then we get

$$\omega^1_\varphi(f^{(k)}, t)_{\varphi^k u} \leq c\, t^\alpha \quad \text{for} \;\; 0 < t \leq t_0$$

and the assertion follows from (1.64) (applied to $f^{(k)}$) and Lemma 1.37. If $\alpha = 1$ and $\alpha_i \notin \{1, \ldots, r\}$ for all $x_i \in (-1,1)$, then $f \in \mathbf{W}(u, k{+}1)$ (see Remark 1.15) and Proposition 1.14 yields the assertion, even if (1.23) is not assumed. ■

Proof of assertion (iii). Similar to the proof of (ii). (Use the first assumption on $f^{(k)}$ and (1.70) with φ^{k+1} replaced by $\varphi^{k+\alpha}$ to prove $\left|\left(\Delta^1_{h\varphi} f^{(k)}\right)(x)\right| \leq c\, h^\alpha \left(\varphi^k u\right)^{-1}(x)$.) ■

Proof of assertion (iv). If $\gamma < 1$, then it easy to prove the assertion with the help of $K^1_\varphi(1/f, t)_u$. (Show that $K^1_\varphi(1/f, t)_u \leq c(f)\, K^1_\varphi(f, t)_u$ for $t \leq t_0(f)$.) One can use the analogue for approximation by trigonometric polynomials together with the characterization of the approximation order with the help of smoothness properties of derivatives to prove the assertion for arbitrary $\gamma \notin \mathbb{N}$ in the trigonometric case. By the well known connection between trigonometric and algebraic approximation, this also gives the assertion for $\mathbf{C}^{\gamma,\delta}$. But, as we said, this approach does only work for $\gamma \notin \mathbb{N}$. For this reason, we will give another proof which is based on the following result: If $\mathbf{C}^{\mathcal{A}}_\infty$ is an approximation space based on $(\mathbf{X}, \{\mathbf{X}_n\}) = (\mathbf{C}, \{\Pi_n\})$, where $\mathcal{A}$ satisfies (in addition to (1.3) and (1.4))

$$K^{-1}a_{2n} \leq a_n \leq K n^c \quad \text{for all} \;\; n \in \mathbb{N} \quad (1 < K \neq K(n),\; 0 < c \neq c(n))\,,$$

then the closure $\operatorname{clos}_{\mathcal{A},\infty}\Pi$ of Π in $\mathbf{C}_\infty^{\mathcal{A}}$ (see Theorem 1.4,(ii)) is inversely closed in $\mathbf{C}$, i.e.,

$$\frac{1}{f} \in \operatorname{clos}_{\mathcal{A},\infty}\Pi \quad \text{for all } \ f \in \operatorname{clos}_{\mathcal{A},\infty}\Pi \ \text{ with } \ f(x) \neq 0, \ x \in [-1,1] \tag{1.71}$$

([AL3, Theorem 2] applied to $S = \big\{\{E_n\} : \lim_{n\to\infty} a_{n+1}E_n = 0\big\}$). Let $\mathcal{A}$ be the sequence from Remark 1.10 and let $f \in \mathbf{C}^{\gamma,\delta}$ with $f(x) \neq 0$ for all $x \in [-1,1]$. Then, for any decreasing sequence $\{\varepsilon_n\}$ with $\lim_{n\to\infty}\varepsilon_n = 0$ and

$$1 = \varepsilon_1 a_1 < \varepsilon_2 a_2 < \varepsilon_3 a_3 < \dots, \quad \lim_{n\to\infty} \varepsilon_n a_n = \infty, \tag{1.72}$$

(1.71) can be applied to f and to $\mathcal{B} = \{\varepsilon_n a_n\}$ instead of $\mathcal{A}$. Hence,

$$\lim_{n\to\infty} \varepsilon_n a_n E_{n-1}^u(1/f) = 0\,. \tag{1.73}$$

Let us assume that $a_n E_{n-1}^u(1/f) \neq O(1)$. Then,

$$\widetilde{\varepsilon}_n = \left[\max_{1\le k\le n} a_k E_{k-1}^u(1/f)\right]^{-1}$$

is a decreasing sequence with limit zero. Moreover, it is clear that

$$\widetilde{\varepsilon}_n = \left[a_n E_{n-1}^u(1/f)\right]^{-1} \quad \text{for all } \ n \in \mathbb{N}', \tag{1.74}$$

where $\mathbb{N}' = \{n_j\}_{j=1}^\infty$ is some subsequence of $\mathbb{N}$ with $n_1 = 1$. Thus,

$$\{\widetilde{\varepsilon}_n a_n\}_{n\in\mathbb{N}'} = \left\{\left[E_{n-1}^u(1/f)\right]^{-1}\right\}_{n\in\mathbb{N}'}$$

is increasing and converges to infinity, since $E_n^u(1/f) \to 0$. Without loss of generality we may assume that $\{\widetilde{\varepsilon}_n a_n\}_{n\in\mathbb{N}'}$ is strictly increasing. Now we define $\{\varepsilon_n\}_{n=1}^\infty$ by

$$\varepsilon_n = \|1/f\|_u\, \widetilde{\varepsilon}_{n_j} \quad \text{for} \quad n_j \le n < n_{j+1} \ \ (j = 1,2,\dots)\,.$$

This sequence is decreasing, converges to zero, and satisfies (1.72). In view of (1.74), $\varepsilon_n a_n E_{n-1}^u(1/f) = \|1/f\|_u$ for all $n \in \mathbb{N}'$, which is in contradiction with (1.73). ■

Proof of assertion (v). One can prove the assertion with the help of the φ–modulus of smoothness. But, if one is not interested in the value of $\gamma(s,u)$, then it is easier to apply Lemma 1.34 and the well known fact that $E_n(g) \le c\,n^{-s}\|g\|_{\mathbf{H}^s}$ for all $g \in \mathbf{H}^s([-1,1])$ (see Lemma 1.37 and (1.50)). In this way we obtain, for some constant $k \in \mathbb{N}$,

$$E_{n^k}^u(f) \le c\,n^{-\min\{1,s\}}\,\|fu\|_{\mathbf{H}^s}\,, \quad n \in \mathbb{N}\,,$$

and, consequently, $E_m^u(f) \le c\,m^{-\min\{1,s\}/k}\|fu\|_{\mathbf{H}^s}$, $m \in \mathbb{N}$. ■

1.4.9 Proof of Proposition 1.19

The following result is well known (see [DL, Theorem 7.3.1]).

Lemma 1.38 *Let $\mathcal{T}_n$ be the space of all trigonometric polynomials of degree $< n$ and let $\mathbf{C}_{2\pi}^s := \mathbf{X}_\infty^{\mathcal{A}}(\{\mathcal{T}_n\})$ with $\mathbf{X} = \{f \in \mathbf{C}(\mathbb{R}) : f(x) = f(x+2\pi)\}$ (endowed with the maximum norm) and $\mathcal{A} = \{n^s\}$. If $s \in (0,1)$, then every $f \in \mathbf{C}_{2\pi}^s$ is Hölder continuous with exponent s and*

$$\sup_{t,\tau \in \mathbb{R},\, t\neq\tau} \frac{|f(t)-f(\tau)|}{|t-\tau|^s} \le c\,\|f\|_{\mathbf{C}_{2\pi}^s}\,, \quad \textit{where } \ c \neq c(f)\,.$$

Proof of assertion (i). From (1.10) and Lemma 1.35 it follows

$$fu \in \mathbf{C}^{\eta,0}\,, \quad \|fu\|_{\eta,0} \le c\,\|f\|_{u,\gamma,\delta}\,, \quad \text{and} \quad (fu)(x_i) = 0 \ \text{ for all } \ i\,,$$

where $\eta = \eta(\gamma,\delta,u) \in (0,1)$ and $c = c(\gamma,\delta,u) > 0$. Clearly, this implies that the function $g(t) = (fu)(\cos t)$ belongs to $\mathbf{C}_{2\pi}^\eta$ (with $\|g\|_{\mathbf{C}_{2\pi}^\eta} \le \|fu\|_{\eta,0}$). By Lemma 1.38 we conclude

$$|(fu)(\cos t) - (fu)(\cos\tau)| \le c\,\|f\|_{u,\gamma,\delta}\,|t-\tau|^\eta\,.$$

Since $\arccos \in \mathbf{H}^{1/2}([-1,1])$, we obtain $\|fu\|_{\mathbf{H}^{\eta/2}} \le c\,\|f\|_{u,\gamma,\delta}$. ∎

Proof of assertion (ii). We apply assertion (viii) of Theorem 1.4 to

$$\mathbf{Y} = \left\{f \in \mathbf{C}^k\big((-1,1)\setminus\{x_i\}\big) : \ f^{(j)} \in \mathbf{C}_{\varphi^j u}\,,\ j = 0,\dots,k\right\}, \quad \|f\|_{\mathbf{Y}} = \sum_{j=0}^k \left\|f^{(j)}\right\|_{\varphi^j u}.$$

First we have to show that $\mathbf{Y}$ is a Banach space. Let $\{f_m\}$ be a Cauchy sequence in $\mathbf{Y}$. Then, $\{f_m\}$ is a Cauchy sequence in $\mathbf{C}^k(I)$ for every closed interval $I \subset (-1,1)\setminus\{x_i\}$. Thus, there exists an $f \in \mathbf{C}^k\big((-1,1)\setminus\{x_i\}\big)$ such that

$$\lim_{m\to\infty} \left\|f^{(j)} - f_m^{(j)}\right\|_{\mathbf{C}(I)} = 0\,,\ \ j = 0,\dots,k\,, \quad \text{for every closed } I \subset (-1,1)\setminus\{x_i\}\,. \tag{1.75}$$

On the other hand, for every $j \in \{0,\dots,k\}$, $\left\{f_m^{(j)}\right\}$ is a Cauchy sequence in $\mathbf{C}_{\varphi^j u}$ and, consequently, convergent in $\mathbf{C}_{\varphi^j u}$. Hence, in (1.75) we can replace $\|\,.\,\|_{\mathbf{C}(I)}$ by $\|\,.\,\|_{\varphi^j u}$, i.e., $f_m \to f$ in $\mathbf{Y}$. Now we mention that, by Proposition 1.14 (applied to $r = 1,\dots,k$), the assumption $\|f_n\|_{\mathbf{Y}} \le c\,a_n\|f_n\|_{\mathbf{X}}$, $f_n \in \mathbf{X}_n = \Pi_n$, is satisfied with $a_n = n^k$ and $\mathbf{X} = \mathbf{C}_u$. Since $\mathbf{C}_u^{\gamma,\delta} = \mathbf{X}_\infty^{\mathcal{AB}}$ with $\mathcal{B}$ corresponding to $(\gamma-k,\delta)$ (see Remark 1.10), we obtain $\mathbf{C}_u^{\gamma,\delta} \subseteq \mathbf{Y}_\infty^{\mathcal{B}}$ and, particularly, $f^{(k)} \in \mathbf{C}_{\varphi^k u}^{\gamma-k,\delta}$. ∎

Proof of assertion (iii). Without loss of generality we may assume that $\delta = 0$. Let $\gamma \in (k,\,k+1]$. In view of assertion (ii), $f^{(k)} \in \mathbf{C}_{\varphi^k u}^{\gamma-k,0}$. This implies

$$\left\|f^{(k)} - P_n\right\|_{\mathbf{C}(J)} \le c\,n^{k-\gamma} \quad \text{for certain } \ P_n \in \Pi_n\,,$$

where J is any closed subinterval of $(-1,1) \setminus \{x_i\}$ such that I is contained in the interior of J. Let $\phi(x) = ax + b$ be the linear function which maps $[-1,1]$ onto J and let $g(t) = f^{(k)}(\phi(\cos t))$. Then we conclude $g \in \mathbf{C}_{2\pi}^{\gamma-k}$ and, in view of Lemma 1.38,

$$\left|f^{(k)}(\phi(\cos t)) - f^{(k)}(\phi(\cos \tau))\right| \le c\,|t-\tau|^s\,, \quad \text{where} \quad s = \begin{cases} \gamma - k & \text{if} \quad \gamma \notin \mathbb{N}\,, \\ \gamma - k - \varepsilon & \text{if} \quad \gamma \in \mathbb{N}\,. \end{cases}$$

Since the inverse of the function $\phi(\cos(\,.\,)) : [0,\pi] \to J$ is continuously differentiable on I, we obtain $f^{(k)} \in \mathbf{H}^s(I)$. ■

Proof of assertion (iv). If $|x_i| < 1$ and $\gamma > \alpha_i$, then

$$\mathbf{C}_u^{\gamma,\delta} \subseteq \mathbf{C}_v^{\gamma-\alpha_i,\delta}\,, \quad \text{where} \quad v(x) = |x - x_i|^{-\alpha_i} u(x)$$

(see assertion (vii) of Theorem 1.11). If further $\gamma - \alpha_i \notin \mathbb{N}$, then $f \in \mathbf{H}^{\gamma-\alpha_i}(I)$ because of assertion (iii). (Remark that $I \subset (-1,1)$ contains no zero of v.)

If $x_i \in \{-1,1\}$ and $(\gamma/2) - \alpha_i \in (k, k + (1/2))$, then, in view of assertion (ii) and Theorem 1.11,

$$f^{(k)} \in \mathbf{C}_{\varphi^k u}^{\gamma-k,\delta} \subseteq \mathbf{C}_v^{\gamma-2k-2\alpha_i,\delta} \subseteq \mathbf{C}_v^{\gamma-2k-2\alpha_i,0}\,,$$

where $\gamma - 2k - 2\alpha_i \in (0,1)$. In the same way as in the proof of assertion (iii) this implies

$$\left|f^{(k)}(\phi(\cos t)) - f^{(k)}(\phi(\cos \tau))\right| \le c\,|t-\tau|^{\gamma-2k-2\alpha_i}\,,$$

where $\phi(x) = ax + b$ is the linear function which maps $[-1,1]$ onto I. Since the inverse of the function $\phi(\cos(\,.\,)) : [0,\pi] \to I$ is Hölder continuous on I with exponent $1/2$, we obtain $f^{(k)} \in \mathbf{H}^{(\gamma/2)-k-\alpha_i}(I)$. ■

1.5 Notes and comments

1.1. The theory of the classical approximation spaces $\mathbf{X}_q^s$ (see Example 1.3) can be found in [P], even in the framework of quasi-normed spaces $\mathbf{X}$ and non-linear subsets $\mathbf{X}_1 \subseteq \mathbf{X}_2 \subseteq \ldots$ satisfying

$$\lambda\,\mathbf{X}_n \subseteq \mathbf{X}_n \quad \text{and} \quad \mathbf{X}_n + \mathbf{X}_m \subseteq \mathbf{X}_{n+m} \quad \text{for all scalars } \lambda \text{ and all } n, m \in \mathbb{N}\,. \tag{1.76}$$

For Banach spaces $\mathbf{X}$ and linear subspaces $\mathbf{X}_n$ approximation spaces with respect to general sequence spaces S (replace the condition $\{a_n(q)E_n(f) \in \mathbf{l}^q$ of Definition 1.1 by $\{E_n(f)\} \in S$) were considered by Brudnyi and Krugliak [BK, Section 4.3.C]. But they concentrate on interpolation properties of these spaces and only a few other results are proved directly without interpolation theory. The most general case of approximation spaces based on quasi-normed spaces $\mathbf{X}$, subsets $\mathbf{X}_n$ satisfying (1.76) (or even a more general condition), and more or less arbitrary sequence spaces S is treated in [AL1]. Applications to the investigation of compactness of subsets of Banach spaces and the inverse closedness of certain commutative algebras are given in [AL2] and [AL3]. All assertions of Theorem 1.4, except (vii), are proved in [AL1] under more general assumptions. Assertion (vii) of Theorem 1.4 and its consequences for the interpolation properties of approximation spaces can be found in [L2]. The special case $q = \infty$ of assertion (vii) of Theorem

1.4 is considered in [BJS] and [Ja]. The proof of Theorem 1.4 given in Subsection 1.4.1 contains several simplifications of the proofs given in [AL1] and [L2], respectively. This is possible because of our more restrictive assumptions on $(\mathbf{X}, \{\mathbf{X}_n\})$ and the introduction of the sequence $\mathcal{A}(q)$, motivated by an idea in [L5]. Numbers $n(j)$ similar to that defined in the proof of Theorem 1.4 are used in another context by Kaljabin and Lizorkin [KL].

1.2. The characterization of the order of the errors of best approximation by algebraic polynomials has a long history. In particular, approximation in the norm of $\mathbf{C}$ is a widely studied subject and we refer to the books [T, Na] and the references given therein for the classical results on this subject. The equivalent characterization of the behavior of best approximation errors with the help of the φ–modulus was first possible for the case of Jacobi weights $u = v^{\rho,\tau}$ (and certain more general weights having no zeros in $(-1,1)$) after the works of Ditzian and Totik on moduli of smoothness and corresponding K–functionals (see [DT]). We also refer to the book of DeVore and Lorentz [DL] in which a big number of classical and more recent results on the connection between polynomial approximation and smoothness properties can be found. The results on approximation in norms with power weights given in Section 1.2 are based on recent works. Proposition 1.6 is surely known, although we have not found it in the literature. The trigonometric counterpart of Theorem 1.13 is well known for the case of uniform approximation. In the general form presented here it was first proved in [L3]. For the proofs of Propositions 1.14, 1.16, and 1.19, Theorem 1.17, and Corollary 1.18 we have mainly used results of Mastroianni, Totik, De Bonis and Russo. The exact references are given in the corresponding subsections of Section 1.4.

1.3. Assertion (i) of Theorem 1.21 was first proved in [AL1] (without stating (1.26); but its proof is, in principle, contained in the proof of [AL1, Theorem 4.3]). The assertions (ii) and (iii) of Theorem 1.21 are well known in the classical case $a_n = n^s$, $b_n = n^t$ (see [P]) and in related cases (see, e.g., [L2, Section 2.2]). For $\mathcal{A}$ and $\mathcal{B}$ satisfying only the assumptions of Theorem 1.21 the assertion (ii) was first given in [L4] (under some slight additional conditions), while the result (iii) is new as far as we know. In [L4] one can also find an alternative proof of Corollary 1.23. In this nice proof estimate (1.26) is used instead of (1.27). Since this yields a considerable simplification if one is only interested in estimates of the left hand side of (1.27) for the case $q = \infty$, let us present the proof of (1.28) as it is given in [L4]: One only has to show that

$$\sum_{m=n}^{\infty} a_m(1)\,(a_m b_m)^{-1} \le c\, b_n^{-1} \quad \text{for all } \; n \in \mathbb{N}\,, \tag{1.77}$$

since then the embedding $\mathbf{X}_\infty^{\mathcal{AB}} \subseteq \mathbf{X}_1^{\mathcal{A}}$ is obvious and the estimate (1.26) leads immediately to (1.28). We may assume that $\{a_n^\varepsilon b_n^{-1}\}_{n=1}^\infty$ is decreasing, since $\{b_n\}$ can be replaced by the equivalent sequence $\{\widetilde{b}_n\}$, $\widetilde{b}_n = a_n^\varepsilon \max\{a_m^{-\varepsilon} b_m : 1 \le m \le n\}$. This assumption, together with the mean value theorem $\ln B/A = \xi^{-1}(B-A)$ $(0 < A < \xi < B)$, leads to the estimate

$$\begin{aligned} b_{m+1}(a_{m+1} - a_m) \quad &\le \quad a_{m+1}b_{m+1} - a_m b_m \le a_{m+1}b_{m+1} \ln \frac{a_{m+1}b_{m+1}}{a_m b_m} \\ &\le \quad a_{m+1}b_{m+1}\left(1 + \frac{1}{\varepsilon}\right) \ln \frac{b_{m+1}}{b_m} \le c\, a_m (b_{m+1} - b_m)\,. \end{aligned}$$

Thus, $a_m(1)\,(a_m b_m)^{-1} \le c\big(b_m^{-1} - b_{m+1}^{-1}\big)$ and (1.77) follows.

Chapter 2

Cauchy singular integral operators on [-1,1]

In the present chapter we study integral operators which are defined with the help of the Cauchy principle value integral

$$(Sf)(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\left(\int_{-1}^{x-\varepsilon}\frac{f(t)}{t-x}\,dt + \int_{x+\varepsilon}^{1}\frac{f(t)}{t-x}\,dt\right),\quad x\in(-1,1)\,.$$

(If $f(\pm 1) = 0$, then $(Sf)(x)$ can also be considered for $x = \pm 1$, where the usual Lebesgue integral is meant in this case.) More precisely, we consider operators of the type

$$A = awI + b_1 S b_2 wI \tag{2.1}$$

(awI denotes the operator of multiplication by aw and b_1Sb_2wI is the composed operator $(b_1I)\,S\,(b_2wI)$), where w is some weight (later we will be more precise), $a, b_1, b_2 : [-1,1] \to \mathbb{C}$ are given bounded functions, and b_2 is supposed to be piecewise $\mathbf{C}^0$. The latter means that

$$b_2(x) = b_2^{(j)}(x) \;\text{ for }\; x\in(\xi_j,\xi_{j+1}),\; j=0,1,\ldots,R, \quad \text{where} \quad b_2^{(j)}\in\mathbf{C}^0$$

and $-1 = \xi_0 < \xi_1 < \ldots < \xi_R < \xi_{R+1} = 1$ ($R\in\mathbb{N}_0$) are given points in which b_2 may have jumps. Instead of $b_2^{(j)}\in\mathbf{C}^0$ we can also write

$$\begin{aligned} & b_2^{(j)} \in \mathbf{C}^0[\xi_j,\xi_{j+1}]\,, \\ & \mathbf{C}^0[c,d] := \left\{ f\in\mathbf{C}[c,d] : \|f\|_{\mathbf{C}^0[c,d]} = \|f\|_{\mathbf{C}[c,d]} + \int_0^1 \omega_{[c,d]}(f,h)\frac{dh}{h} < \infty \right\} \end{aligned} \tag{2.2}$$

$\big($compare Theorem 1.13; $\omega_{[c,d]}(f,h) := \sup\big\{ |f(x)-f(y)| : x,y\in[c,d]\,,\,|x-y|\le h\big\}\big)$. Indeed, for $f\in\mathbf{C}[c,d]$ ($[c,d]\subseteq[-1,1]$),

$$f\in\mathbf{C}^0[c,d] \;\text{ if and only if }\; \widetilde{f}\in\mathbf{C}^0\,,\quad \widetilde{f}(x) = \begin{cases} f(x)\,, & x\in[c,d]\,, \\ f(d)\,, & x>d\,, \\ f(c)\,, & x<c\,, \end{cases} \tag{2.3}$$

where $\|f\|_{\mathbf{C}^0[c,d]} \sim \big\|\widetilde{f}\big\|_0$. $\big($Use that $\omega_{[c,d]}(f,h) = \omega\big(\widetilde{f},h\big)$.$\big)$

The weight w in (2.1) is a fixed integrable power weight, i.e.,

$$w(x) = \prod_{i=1}^{L} |x - z_i|^{\mu_i} \quad \text{with} \quad -1 \le z_1 < z_2 < \ldots < z_L \le 1 \;\text{ and }\; \mu_i > -1$$

($L \in \mathbb{N}_0$, $w = 1$ for $L = 0$). We will show that, for any choice of power weights

$$u(x) = \prod_{i=1}^{N} |x - x_i|^{\alpha_i} \quad \text{and} \quad v(x) = \prod_{j=1}^{M} |x - y_j|^{\beta_j} \tag{2.4}$$

($N, M \in \mathbb{N}_0$, $-1 \le x_1 < x_2 < \ldots < x_N \le 1$, $-1 \le y_1 < y_2 < \ldots < y_M \le 1$) with

$$w = \frac{u}{v}\,, \quad \alpha_i > 0 \;\text{ for all } i \quad \text{and} \quad 0 < \beta_j < 1 \;\text{ for all } j\,, \tag{2.5}$$

the operator A maps certain subspaces of $\mathbf{B}_u$ (see Example 1.24 for the definition of $\mathbf{B}_u$) into subspaces of $\mathbf{B}_{\widetilde{v}}$, where $\widetilde{v}$ is some modification of v.

2.1 Cauchy singular integral operators on $\mathbf{C}_u^0$

Take the above notation and assumptions and define, for finite subsets $\mathcal{M}$ of $[-1, 1]$,

$$\big(v[\mathcal{M}]\big)(x) = v(x) \left(1 + \sum_{\xi \in \mathcal{M} \setminus \{y_1, y_2, \ldots, y_M\}} \big| \ln |x - \xi| \, \big| \right)^{-1} . \tag{2.6}$$

Thus, $v[\mathcal{M}]$ is a "logarithmic modification" of v which vanishes on $\mathcal{M}$. If v is already vanishing on $\mathcal{M}$, i.e., $\mathcal{M} \subseteq \{y_1, y_2, \ldots, y_M\}$, then $v[\mathcal{M}] = v$.

Theorem 2.1 *$A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_{v[\mathcal{M}]})$ with $\mathcal{M} = \{\xi_0, \xi_1, \ldots, \xi_{R+1}\} \setminus \{x_1, x_2, \ldots, x_N\}$ and*

$$\|A\|_{\mathbf{C}_u^0 \to \mathbf{B}_{v[\mathcal{M}]}} \le c \left(\|a\| + \|b_1\| \max \left\{ \|b_2^{(0)}\|_0, \|b_2^{(1)}\|_0, \ldots, \|b_2^{(R)}\|_0 \right\} \right) \tag{2.7}$$

($c \ne c(a, b_1, b_2)$). Particularly, for every $f \in \mathbf{C}_u^0$, the functions $b_1 S b_2 w f$ and awf are well-defined on $\operatorname{supp} v[\mathcal{M}]$*, where the latter is meant in the sense $awf = av^{-1} \cdot uf$, uf considered as continuous function on $[-1, 1]$.*

Remark 2.2 *We mention that, for $\mathcal{M}$ from Theorem 2.1, $v[\mathcal{M}] = v$ if and only if $(uv)(\xi_i) = 0$ for all $i = 0, 1, \ldots, R+1$.*

In view of Theorem 1.13, we have

$$\|f\|_{u,0} \sim \|fu\|_0 \;\text{ for all }\; f \in \mathbf{C}_u^0 = \{f \in \mathbf{C}_u : fu \in \mathbf{C}^0,\, (fu)(x_i) = 0 \text{ for all } i\}. \tag{2.8}$$

One may ask whether $Af \in \mathbf{B}_{v[\mathcal{M}]}$ for functions f for which fu is a $\mathbf{C}^0$–function (or even piecewise $\mathbf{C}^0$–function) which does not vanish on $\{x_1, x_2, \ldots, x_N\}$ but on another (possibly empty) fixed set $T = \{t_1, t_2, \ldots, t_S\} \subseteq [-1,1]$ ($S \in \mathbb{N}_0$). The answer to this question is contained in Theorem 2.1. Indeed, if

$$\begin{array}{l} fu : \operatorname{supp}_* u \to \mathbb{C} \text{ possesses a piecewise continuous extension } \sum\limits_{j=0}^{R} g_j\, \chi_j \text{ on } [-1,1], \\ \text{where } g_j \in \mathbf{C}^0,\ g_j = 0 \text{ on } T \text{ for all } j, \text{ and } \chi_j \text{ are the characteristic functions of} \\ \text{the disjoint subintervals } I_j \text{ of a partition of } [-1,1] \text{ with the breakpoints } \left\{\xi_j\right\}_{j=0}^{R+1}, \end{array} \tag{2.9}$$

then we may apply Theorem 2.1 to each addend of the decomposition

$$Swb_2 f = \sum_{j=0}^{R} \left(S\, \frac{u_T}{v}\, b_2\, \chi_j I \right) \left(\frac{g_j}{u_T} \right), \quad u_T(x) = \prod_{i=1}^{S} |x - t_i| \,,$$

taking into account that, by (2.8), $\dfrac{g_j}{u_T} \in \mathbf{C}^0_{u_T}$. We obtain the following result.

Corollary 2.3 *If f satisfies (2.9), then $Af \in \mathbf{B}_{v[\mathcal{M}]}$ with $\mathcal{M} = \{\xi_0, \xi_1, \ldots, \xi_{R+1}\} \setminus T$ and*

$$\|Af\|_{v[\mathcal{M}]} \le c \max\left\{ \|g_0\|_0 \,,\, \|g_1\|_0 \,,\, \ldots \,,\, \|g_R\|_0 \right\},$$

where c is independent of f and of the choice of the functions g_j. (The part awf of Af is understood in the sense $awf = av^{-1} \cdot uf$, uf the extension from (2.9).)

Other important consequences of Theorem 2.1 are given in Example 1.24. We repeat estimate (1.29):

Corollary 2.4 *Take $\mathcal{M}$ from Theorem 2.1. Then,*

$$\|Ap_n\|_{v[\mathcal{M}]} \le c\, \|p_n\|_u \ln(n+1) \quad \text{for all } p_n \in \Pi_n \text{ and all } n \in \mathbb{N}\,, \tag{2.10}$$

where $c \ne c(n, p_n)$.

2.2 Cauchy singular integral operators on $\mathbf{C}_u^{\gamma,\delta}$

We consider again an operator of the type (2.1) (under the same assumptions on the coefficient functions a, b_1, b_2) in pairs of spaces which are defined with the help of weights u and v satisfying (2.4) and (2.5). But now we restrict A to the space $\mathbf{C}_u^{\gamma,\delta}$. (Remember that, by Theorem 2.1, A is already defined on the bigger space $\mathbf{C}_u^0$.) Since we have only $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_{v[\mathcal{M}]})$ and not $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{C}^0_{v[\mathcal{M}]})$, we do not expect that A maps $\mathbf{C}_u^{\gamma,\delta}$ into $\mathbf{C}^{\gamma,\delta}_{v[\mathcal{M}]}$ (although such a result is known for very special types of operators and weights; see [MRT]). For this reason, we only ask for the validity of $A \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}^{\gamma,\delta-1}_{v[\mathcal{M}]}\big)$. ($\mathcal{M}$ from Theorem 2.1). We restrict ourself to certain special cases which are of interest in our later applications. The simplest special case is that of an operator which maps polynomials into polynomials, since then we can apply the last assertion of Example 1.24. Because of its importance in later parts of this chapter, let us formulate this assertion once again.

Proposition 2.5 *If there is a constant $k \in \mathbb{N}$ such that*

$$A(\Pi_n) \subseteq \Pi_{kn} \quad \textit{for all} \;\; n \in \mathbb{N}\,,$$

then $A \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_{v[\mathcal{M}]}^{\gamma,\delta-1}\big)$ for all $\gamma > 0$ and all $\delta \in \mathbb{R}$, where $\mathcal{M}$ is given in Theorem 2.1.

If $v[\mathcal{M}] \neq v$, then we have the problem that no description of $\mathbf{C}_{v[\mathcal{M}]}^{\gamma,\delta-1}$ in terms of smoothness properties of its elements is given in Section 1.2. Instead of giving such a description, we only mention here that $\mathbf{C}_{v[\mathcal{M}]}^{\gamma,\delta-1}$ is not much bigger than $\mathbf{C}_v^{\gamma,\delta-1}$. More precisely, the following holds true.

Lemma 2.6 *Let $\mathcal{M}$ be a finite subset of $[-1,1]$ and let $\gamma > 0$, $\delta \in \mathbb{R}$. Then,*

$$\mathbf{C}_{v[\mathcal{M}]}^{\gamma,\delta} \subseteq \mathbf{C}_v^{\gamma,\delta-1} \quad (\textit{continuous embedding})\,,$$

where the values of $f \in \mathbf{C}_{v[\mathcal{M}]}^{\gamma,\delta}$ in $x \in \operatorname{supp} v \setminus \operatorname{supp} v[\mathcal{M}]$ are understood in the sense of limits.

This is a consequence of assertion (viii),(b) of Theorem 1.4, since

$$\|p_n\|_v \leq c\, \|p_n\|_{v[\mathcal{M}]} \ln(n+1) \quad \text{for} \;\; p_n \in \Pi_n \;\; \text{and} \;\; n \in \mathbb{N} \quad (c \neq c(n, p_n))$$

(see (1.69)). Lemma 2.6 and Proposition 2.5 imply the following.

Corollary 2.7 *Let $k \in \mathbb{N}$, $\gamma > 0$, $\delta \in \mathbb{R}$ be constant. Then,*

$$A \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-2}\big) \quad \textit{if} \quad A(\Pi_n) \subseteq \Pi_{kn} \;\; \textit{for all} \;\; n\,. \tag{2.11}$$

(The domain of definition of Af, $f \in \mathbf{C}_u^{\gamma,\delta}$, is given in Theorem 2.1. For $x \in \operatorname{supp} v$ not belonging to this domain $(Af)(x)$ is defined in the sense of a limit.)

Of course, $A(\Pi_n) \subseteq \Pi_{kn}$ is a very restrictive assumption. However, in every case one can decompose A into two parts one of which maps polynomials into polynomials and the other is a multiplication operator. For the sake of simplicity, we give the following additional assumptions:

$$0 < \alpha_i < 1 \;\; \text{for all} \;\; i = 1, \ldots, N\,, \quad (uv)(\xi_i) = 0 \;\; \text{for all} \;\; i = 0, \ldots, R+1\,, \tag{2.12}$$

$$w_-^{-1} \in \mathbf{L}^1(-1,1)\,, \;\; \text{where} \;\; w_-(x) := \prod_{i=1}^{N} |x - x_i|^{1-\alpha_i} \prod_{j=1}^{M} |x - y_j|^{\beta_j}\,, \tag{2.13}$$

$$b_1 = 1\,; \quad b_2 =: b\,. \tag{2.14}$$

Now we define $p_u(x) = \prod_{i=1}^{N} (x - x_i)$ and consider the following decomposition of $(Af)(x)$,

$$(Af)(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(t)\, p_u(t) - f(x)\, p_u(x)}{t - x}\, b(t) \frac{w(t)}{p_u(t)}\, dt + f(x)\, p_u(x)\, (A p_u^{-1})(x)\,. \tag{2.15}$$

Obviously, the first addend of (2.15) belongs to $\Pi_{n+\deg p_u-1}$ if $f \in \Pi_n$. The second addend is the image $(p_u A p_u^{-1}) \cdot f$ of the operator of multiplication by

$$p_u A p_u^{-1} = a\,w + p_u\, S\,\widetilde{b}\,w_-^{-1}\,, \quad \widetilde{b}(x) = b(x)\prod_{i=1}^{N} \operatorname{sign}(x - x_i)\,.$$

In view of Theorem 2.1 (applied to 1 and w_- instead of u and v), $S\,\widetilde{b}\,w_-^{-1} \in \mathbf{B}_{w_-}$ and, consequently, $p_u A p_u^{-1} \in \mathbf{B}_{1/w}$. Thus, $(p_u A p_u^{-1}) \cdot I \in \mathcal{L}(\mathbf{B}_u, \mathbf{B}_v) \subseteq \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_v)$. Since also $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_v)$ (by Theorem 2.1), we obtain that the first addend of (2.15) defines a bounded operator from $\mathbf{C}_u^0$ into $\mathbf{B}_v$ which maps Π_n into $\Pi_{n+\deg p_u-1}$. Hence, this operator belongs to $\mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}\big)$ for all $\gamma > 0$, $\delta \in \mathbb{R}$. Let us summarize.

Proposition 2.8 *Let $\gamma > 0$, $\delta \in \mathbb{R}$, and let the additional conditions (2.12)–(2.14) be satisfied. Then, $A \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}\big)$ if and only if $(p_u A p_u^{-1}) \cdot I \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}\big)$.*

Let us mention that, in general, the property $(p_u A p_u^{-1}) \cdot I \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}\big)$ can only be expected if a and b are smooth enough. We refer to [L3] for more details and for generalizations of Proposition 2.8. Summing up, we can say that the pair $\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}\big)$ or even the pair $\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-2}\big)$ is not appropriate to study the mapping properties of general Cauchy singular integral operators with discontinuous coefficients a and b. However, we hope that in the future Theorem 2.1 can be used as a basic tool in the study of weighted uniform convergence of approximation methods for integral equations in which operators with piecewise continuous coefficients appear. In the present paper we will not proceed in such a general setting. Instead of this we will consider operators A with Hölder continuous coefficients in all of what follows.

2.3 The general set of solutions of certain integral equations

In the present section we describe the set of solutions of an equation of the type

$$(aI + SbI)f = g\,. \tag{2.16}$$

Here we suppose that a and b are given real-valued and Hölder continuous functions on $[-1, 1]$ with

$$r(x) := \sqrt{a^2(x) + b^2(x)} > 0 \quad \text{for all } \ x \in [-1, 1]\,. \tag{2.17}$$

It is well known that, under these conditions on a and b, there exists an at least one-sided inverse B of the operator $A = aI + SbI$ in the space $\mathcal{L}\big(\mathbf{L}^p(-1, 1)\big)$ if $p > 1$ is sufficiently small (see [GK, Theorem IX.5.1]). This leads to integral representations of all solutions of (2.16), since B is an integral operator which can be given explicitly (as we will see later). By "all solutions" we mean the solutions which belong to $\bigcup_{p>1} \mathbf{L}^p(-1, 1)$. The aim of this section is just to give the integral representations of these solutions, without deep discussions of their smoothness properties (since this will be done later). To stay within the framework of this paper, we will not present the $\mathbf{L}^p$-theory. Instead of this we consider the equation (2.16) in an appropriate linear space $\mathbf{H}$ of functions which are continuous with exception of a finite number of possible singularities. By "appropriate" we mean that $\mathbf{H}$ must have the following property to ensure that no $\mathbf{L}^p$-solution is lost:

$$\text{If } g \in \mathbf{H} \text{ and if } f \in \mathbf{L}^p(-1,1)\ (p>1) \text{ is a solution of (2.16), then } f \in \mathbf{H}. \tag{2.18}$$

To define $\mathbf{H}$, let us first fix the set $\mathcal{S}$ of the possible singularity points of its elements,

$$\mathcal{S} = \{x_0, x_1, \ldots, x_{N+1}\}\,, \quad -1 = x_0 < x_1 < \ldots < x_{N+1} = 1\,.$$

Now, $\mathbf{H}$ is the space of all locally Hölder continuous functions on $[-1,1]\setminus\mathcal{S}$ with possible singularities of the type $O(|x-x_j|^{-\tau_j})$, $\tau_j < 1$, in the points $x_j \in \mathcal{S}$, shortly

$$\mathbf{H} = \mathbf{H}_{\rm loc}(\mathcal{S})\,.$$

More precisely, we say that $f : [-1,1]\setminus\mathcal{S} \to \mathbb{C}$ belongs to $\mathbf{H}_{\rm loc}(\mathcal{S})$ if there exists a power weight $u = u_f$ of the form

$$u(x) = \prod_{j=0}^{N+1} |x-x_j|^{\tau_j} \quad \text{with} \quad 0 \le \tau_j < 1 \ \text{ for all } j \tag{2.19}$$

such that fu possesses a Hölder continuous extension on $[-1,1]$. It is important that, in every case, the points -1 and 1 are admitted as singularity points. This is adapted to the fact that, in general, solutions of (2.16) may have singularities in ± 1 even if the right hand side has no singularity. For example, the solutions f of $Sf = 1$ are given by

$$f(x) = \frac{x+c}{\sqrt{1-x^2}}\,, \quad c \in \mathbb{C} \ \text{ arbitrary} \tag{2.20}$$

([PS, Theorem 9.17 and formula 9.15.(2)]). This example also shows why we have not specified the weight u in the definition of $\mathbf{H}_{\rm loc}(\mathcal{S})$: If we would suppose $f \in \mathbf{C}_u$, where u is some fixed weight, then it might happen that we do not find all solutions. For example, $Sf = 1$ has infinitely many solutions in $\mathbf{H}_{\rm loc}(\{-1,1\})$, but in $\mathbf{C}_u$ with $u = v^{\rho,\tau}$, where

$$v^{\rho,\tau}(x) = (1-x)^\rho (1+x)^\tau \quad \text{and} \quad 0 \le \rho \le \frac{1}{2}\,, \ \ 0 \le \tau < \frac{1}{2}\,,$$

there exists only one solution if $\rho = 1/2$ ($c = 1$ in (2.20)) and no solution if $\rho < 1/2$.

Although we do not study (2.16) in $\mathbf{L}^p$, it is important to know that (2.18) is true.

Proposition 2.9 *The property (2.18) is satisfied for* $\mathbf{H} = \mathbf{H}_{\rm loc}(\mathcal{S})$.

Now, motivated by Proposition 2.9, we study (2.16) as an operator equation in $\mathbf{H}_{\rm loc}(\mathcal{S})$, although this is only a linear space and not a Banach space. First we remark that, for $f \in \mathbf{H}_{\rm loc}(\mathcal{S})$, $Af = af + Sbf$ is a well-defined function on $[-1,1]\setminus\mathcal{S}$, since $f \in \mathbf{C}_u^0$ for some $u = u_f$ of the form (2.19) (this follows from Theorem 1.13) and, consequently, Theorem 2.1 can be applied. To prove that even $Af \in \mathbf{H}_{\rm loc}(\mathcal{S})$, we need the following well known result (see [PS, Theorem 9.9 and Proposition 9.7]).

Lemma 2.10 *Let $A = aI + SbI$. There is a constant $\varkappa_0 \in \mathbb{Z}$ and a weight σ_0 of the form*

$$\sigma_0(x) = v^{\alpha_0,\beta_0}(x)\, h(x)\,, \quad -1 < \alpha_0, \beta_0 \le 0\,,$$

where h is a positive and Hölder continuous function on $[-1, 1]$, such that

$$A(\sigma_0\Pi_n) \subseteq \Pi_{n-\varkappa_0} \quad \textit{for all} \;\; n \in \mathbb{N}\,.$$

(For $k = n - \varkappa_0 \le 0$ we set $\Pi_k = \{0\}$.)

In view of Proposition 2.5, we obtain

$$A\sigma_0 I \in \mathcal{L}\big(\mathbf{C}^{\gamma,\delta}_{u_0}, \mathbf{C}^{\gamma,\delta-1}_{u}\big)\,, \quad u_0 := v^{\alpha_0,\beta_0}u \tag{2.21}$$

for all $\gamma > 0$, $\delta \in \mathbb{R}$, and all weights u of the form (2.19) which satisfy

$$\tau_0 > -\beta_0 \quad \text{and} \quad \tau_{N+1} > -\alpha_0\,.$$

Now, if $f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$, then there exists a weight u of the above type such that fu is Hölder continuous and vanishes in the zeros of u. In view of Corollary 1.18, (v), this implies $\sigma_0^{-1}f \in \mathbf{C}^{\gamma,0}_{u_0}$ for some $\gamma > 0$ and, consequently, $Af = A\sigma_0\, \sigma_0^{-1}f \in \mathbf{C}^{\gamma,-1}_{u}$. Together with assertion (i) of Proposition 1.19 we obtain

$$A \in \mathbb{L}\big(\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big)\,. \tag{2.22}$$

The proof of (2.22) is an example of a general principle: If one has a certain mapping property of the operator A restricted on $\sigma_0\Pi$, then one can use (2.21) and results on approximation spaces to obtain a similar property for A on $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$. Particularly, if we can find a one-sided inverse of the restricted operator $A \in \mathbb{L}(\sigma_0\Pi, \Pi)$ (having similar properties than A), then we obtain also a one-sided inverse of $A \in \mathbb{L}\big(\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big)$. Using this principle, we will obtain a nice theorem about the representation of all solutions of (2.16). In the present section we will only state this theorem. The details of the proof are given later. Here we only mention that the properties of A in $\mathbb{L}(\sigma_0\Pi, \Pi)$ are well-known. For example, Lemma 2.10 reflects a small part of these properties. Clearly, the weight σ_0 from Lemma 2.10 will play an important role in the mentioned theorem. So we shall first give the precise definition of σ_0. For this aim, choose an argument function

$$G(x) = \frac{1}{\pi}\arg\big[a(x) - \mathrm{i}\, b(x)\big] \quad \text{such that} \quad G \in \mathbf{C}[-1, 1]\,. \tag{2.23}$$

(It is easy to see that the difference of two such functions G_1 and G_2 is an even integer. For this reason, the following definition of σ_0 will not depend on the choice of G.) Let $\lceil G(1)\rceil = \min\{k \in \mathbb{Z} : k \ge G(1)\}$, $\lfloor G(-1)\rfloor = \max\{k \in \mathbb{Z} : k \le G(-1)\}$, and set

$$\alpha_0 = G(1) - \lceil G(1)\rceil\,, \quad \beta_0 = \lfloor G(-1)\rfloor - G(-1)\,.$$

Moreover, let P_G be the linear interpolation polynomial of G with respect to the knots -1 and 1, i.e., $P_G(x) = \dfrac{1-x}{2}\,G(-1) + \dfrac{1+x}{2}\,G(1)$, and define

$$h(t) = \frac{\big[\mathrm{e}^{-2}(1-t)^{1-t}(1+t)^{1+t}\big]^{(G(-1)-G(1))/2}}{r(t)}\exp\int_{-1}^{1}\frac{G(x) - P_G(x)}{x - t}\,dx\,. \tag{2.24}$$

(Of course, the last integral is meant in the sense of the Cauchy principle value). Then, h is a positive Hölder continuous function on $[-1,1]$ and σ_0 is given by

$$\sigma_0(t) = v^{\alpha_0,\beta_0}(t)\,h(t)$$

(see [PS, Proposition 9.7 and its proof]). Further, we introduce the notation

$$\varkappa_0 = \lceil G(1) \rceil - \lfloor G(-1) \rfloor \quad \text{and} \quad \mu_0(t) = \frac{1}{\sigma_0(t)\,r^2(t)}\,. \tag{2.25}$$

Theorem 2.11 *Let the right hand side g of (2.16) belong to $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$. The operators*

$$A = aI + SbI \quad \text{and} \quad B = \sigma_0(aI - SbI)\mu_0 I$$

are linear in $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ with

$$\ker A = \sigma_0 \Pi_{\varkappa_0}\,, \quad \ker B = \Pi_{-\varkappa_0}\,, \quad BA = I \ \text{ if } \varkappa_0 \le 0\,, \quad AB = I \ \text{ if } \varkappa_0 \ge 0\,.$$

If $\varkappa_0 \ge 0$, then all solutions $f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ of (2.16) are given by

$$f = Bg + \sigma_0\,p\,, \quad p \in \Pi_{\varkappa_0} \ \text{ arbitrary}\,.$$

If $\varkappa_0 < 0$, then (2.16) has either no solution in $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ (if $g \notin \operatorname{im} A$) or only one, namely

$$f = Bg \quad \text{if } \ g \in \operatorname{im} A\,.$$

If b has at most finitely many zeros in $[-1,1]$ and if there exists a representation $b = \widetilde{b}q$, where q is a polynomial such that all its zeros in $[-1,1]$ are also zeros of b, and $\widetilde{b}$ is some nonnegative function with $\widetilde{b} \in \mathbf{H}_{\mathrm{loc}}(\widetilde{\mathcal{S}})$ for some finite set $\widetilde{\mathcal{S}} \subseteq (-1,1)$ with $\widetilde{\mathcal{S}} \cap \mathcal{S} = \emptyset$, then

$$\operatorname{im} A = \left\{ g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) : \int_{-1}^{1} g(t)\,p(t)\,b(t)\,\mu_0(t)\,dt = 0 \ \text{ for all } \ p \in \Pi_{-\varkappa_0} \right\}. \tag{2.26}$$

(Of course, in the case $\varkappa_0 \ge 0$, (2.26) is also true if the additional condition on b is not satisfied.)

We mention that, in every case, $A(BA - I) = 0$ and $B(AB - I) = 0$. Together with the formulas for the kernels of A and B we obtain the following.

Corollary 2.12 *$BA - I$ maps $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ into $\sigma_0\Pi_{\varkappa_0}$ and $AB - I$ maps $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ into $\Pi_{-\varkappa_0}$. Hence, in every case BA and AB are of the form identity plus finite rank operator.*

Later we will consider equations of the type $(A+K)f = g$, where K is some smoothing integral operator. If BK is a compact operator in some Banach space $\mathbf{X} \subseteq \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ which contains $\sigma_0\Pi_{\varkappa_0}$ and if $Bg \in \mathbf{X}$, then Corollary 2.12 shows that multiplying B from the left to both sides of the equation $(A+K)f = g$ yields a Fredholm equation of the form $(I+H)f = \widetilde{g}$, where $H = (BA - I) + BK$ is a compact operator in $\mathbf{X}$.

2.4 Solutions which vanish in -1 or in $+1$

Let us consider again the equation (2.16). (We take the same assumptions and notation as in the preceding section.) Depending on the practical problem which is modelled by (2.16) (or by $(A+K)f=g$) one could be interested in solutions $f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ which vanish in the endpoints of the interval $[-1,1]$ or in one of the endpoints. In the present section we will discuss the questions of existence and uniqueness of such solutions. More precisely, we fix $k,l \in \{0,1\}$ and ask for the set of solutions belonging to the space

$$\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S}) := \Big\{ f \in \mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S}) : \, f(1)=0 \text{ if } k=1 \text{ and } f(-1)=0 \text{ if } l=1 \Big\}\,, \quad \text{where}$$

$$\mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S}) := \big\{ f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) : \, f \text{ is Hölder continuous on } [-1,-1+l\,\varepsilon) \cup (1-k\,\varepsilon,1] \big\}\,.$$

(Of course, $\varepsilon > 0$ has to be chosen in such a way that $(-1,-1+\varepsilon] \cup [1-\varepsilon,1)$ contains no element of $\mathcal{S}$.) We suppose that the right hand side g of (2.16) belongs to $\mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ and that

$$k=0 \text{ if } \alpha_0 = 0 \quad \text{and} \quad l = 0 \text{ if } \beta_0 = 0\,. \tag{2.27}$$

Theorem 2.13 *Let $k,l \in \{0,1\}$ satisfy (2.27) and let $g \in \mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$. Further, set*

$$\varkappa = \varkappa_0 - k - l\,, \quad \sigma = \sigma_0\, v^{k,l}\,, \quad \mu = \mu_0\, v^{-k,-l}\,, \quad B_{k,l} = \sigma(aI - SbI)\mu I\,.$$

If $\varkappa \geq 0$, then all solutions $f \in \mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ of (2.16) belong to $\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ and they are given by

$$f = B_{k,l}\, g + \sigma\, p\,, \quad p \in \Pi_\varkappa \text{ arbitrary}\,. \tag{2.28}$$

If $\varkappa < 0$, then (2.16) has either no solution in $\mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ or only one, namely

$$f = B_{k,l}\, g \quad \text{if } g \in A\big(\mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})\big)\,.$$

In the last case f is an element of $\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ and g must satisfy

$$\int_{-1}^{1} g(t)\, p(t)\, b(t)\, \mu(t)\, dt = 0 \quad \text{for all} \quad p \in \Pi_{-\varkappa}\,. \tag{2.29}$$

If $\varkappa_0 \geq 0$ or if b satisfies the additional assumptions given in Theorem 2.11, then (2.29) is necessary and sufficient for the existence of a solution $f \in \mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ of (2.16).

On the first view it is surprising that the operator $B_{k,l}$ appears instead of the operator B from Theorem 2.11. But if we write

$$B_{k,l} = B + \sigma_0(Sv^{k,l}I - v^{k,l}S)b\mu I\,, \tag{2.30}$$

then we see that B and $B_{k,l}$ are closely related and it is easy to show that, if $g \in A(\mathbf{H}_{\mathrm{loc}}(\mathcal{S})) \cap \mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ satisfies (2.29), $B_{k,l}\, g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ is a solution of (2.16). Indeed, $g \in \mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ implies $\mu g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$, which shows that $B_{k,l}\, g$ is well-defined. Moreover, the kernel $[\pi(t-x)]^{-1}[v^{k,l}(t) - v^{k,l}(x)]$ of $Sv^{k,l}I - v^{k,l}S$ is a polynomial of degree $k+l-1$ in both variables (namely, $\pm\pi^{-1}(x+t)^{k+l-1}$ if $k+l>0$) and, consequently,

$$\sigma_0(Sv^{k,l}I - v^{k,l}S)b\mu g \in \sigma_0\Pi_{k+l}\,. \tag{2.31}$$

If $\varkappa \geq 0$, then $\sigma_0\Pi_{k+l} \subseteq \sigma_0\Pi_{\varkappa_0} = \ker A$ and, hence, $AB_{k,l}\, g = ABg = g$ by Theorem 2.11. If $\varkappa < 0 < \varkappa_0$, then $k = l = \varkappa_0 = -\varkappa = 1$ and (2.29) shows that the linear part of $(Sv^{k,l}I - v^{k,l}S)b\mu g$ is zero, i.e., that the left hand side of (2.31) is an element of $\sigma_0\Pi_1 = \ker A$. If $\varkappa_0 \leq 0$, then $k + l \leq -\varkappa$ and the left hand side of (2.31) vanishes because of (2.29). Thus, in every case $AB_{k,l}\, g = ABg$ and Theorem 2.11 shows that $f = B_{k,l}\, g$ is a solution of (2.16) if $g \in A(\mathbf{H}_{\mathrm{loc}}(\mathcal{S})) \cap \mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ satisfies (2.29). The proof of the remaining assertions of Theorem 2.13, particularly the proof of

$$B_{k,l} \in \mathbb{L}\big(\mathbf{H}^{k,l}_{\mathrm{loc}}(\mathcal{S}), \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})\big)\,, \tag{2.32}$$

is given later. In this proof we will even see that the set of solutions f of (2.16) becomes not larger if we look for $f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ instead of $f \in \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$. More precisely, one can show that

$$A, B_{k,l} \in \mathbb{L}\big(\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})\big)\,, \tag{2.33}$$

and that the following holds true. (We mention that $f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ belongs to $v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ if and only if $v^{-\alpha_0,-\beta_0}f$ is Hölder continuous on the set $[-1, -1 + l\,\varepsilon) \cup (1 - k\,\varepsilon, 1]$, where $(v^{-\alpha_0,-\beta_0}f)(\pm 1) = 0$ if ± 1 belongs to this set.)

Remark 2.14 *Let (2.27) be satisfied and let* $g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$. *If* $\varkappa \geq 0$, *then all solutions* $f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ *of (2.16) are given by (2.28), i.e.,*

$$AB_{k,l} = I \quad \text{on} \quad \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S}) \quad \text{and} \quad \ker A \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S}) = \sigma\Pi_{\varkappa}\,.$$

If $\varkappa \leq 0$, *then* $\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ *contains at most one solution* f *of (2.16) and, if this solution exists, then it is given by* $f = B_{k,l}\, g$, *i.e.,*

$$B_{k,l}A = I \quad \text{on} \quad \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})\,.$$

A necessary condition for the existence of a solution in $\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$ *is given by (2.29). This condition is also sufficient if* $\varkappa_0 \geq 0$ *or if* b *satisfies the additional assumptions given in Theorem 2.11.*

Example 2.15 *Let us consider the equation* $Sf = 1$. *We have* $G \equiv -1/2$ *and, hence,*

$$\alpha_0 = \beta_0 = -\frac{1}{2} \quad \text{and} \quad \varkappa_0 = 1\,.$$

Theorem 2.13 asserts that there exists exactly one solution which vanishes in 1 $(k = 1, l = 0, \varkappa = 0)$ *and also exactly one solution which vanishes in* -1 $(k = 0, l = 1, \varkappa = 0)$. *Clearly, we obtain the same result if we look at the known set of all solutions of* $Sf = 1$ *given by formula (2.20). Namely,*

$$f(x) = -\sqrt{\frac{1-x}{1+x}} \quad \text{and} \quad f(x) = \sqrt{\frac{1+x}{1-x}} \tag{2.34}$$

are the unique solutions which vanish in 1 *and* -1, *respectively. The same solutions are obtained if* $f(x)\sqrt{1-x^2}$ *shall vanish in* 1 *and* -1, *respectively. This is in accordance with Remark 2.14. None of the functions (2.34) (or their products with* $\sqrt{1-x^2}$*) vanishes in* *both points* -1 *and* 1. *This corresponds to the case* $k = l = 1$, $\varkappa = -1$ *of Theorem 2.13 (Remark 2.14): For* $g = b = 1$ *the condition (2.29) is not satisfied. Let us replace* $g = 1$ *by* $g(x) = x$. *Then, (2.29) is satisfied and*

$$f(x) = -\sqrt{1-x^2} \quad \textit{solves} \quad (Sf)(x) = x$$

([PS, Formula 9.15.(2)]). By Theorem 2.13 (Remark 2.14), this is the only solution with $f(-1) = f(1) = 0$ *((*$(\varphi f)(-1) = (\varphi f)(1) = 0$, $\varphi(x) = \sqrt{1-x^2}$*). This becomes also clear if we take into account that* $\ker S = \operatorname{span}\{(1-x^2)^{-1/2}\}$ *(see (2.20)).*

2.5 An important special case

For the explicit determination of solutions f of $Af = g$ Theorems 2.11 and 2.13 are only useful if the weights $\sigma_0 = v^{\alpha_0,\beta_0}h$ and $\mu_0 = 1/(r^2\sigma_0)$ which appear in $B = \sigma_0(aI - SbI)\mu_0 I$ can be computed. This means that the integral

$$\int_{-1}^{1} \frac{G(x) - P_G(x)}{x - t}\, dx$$

on the right hand side of (2.24) can be determined. Let us study the simplest (but, as we will see later, very important) case, where $G(x)$ is a linear function and the coefficient b of $A = aI + SbI$ can be taken identically 1. In this case we may assume that $-1 < G(x) < 0$ for all $x \in [-1, 1]$. Let us define $G(x)$ in dependence of two numbers $\alpha_0, \beta_0 \in (-1, 0)$ such that $G(1) = \alpha_0$ and $G(-1) = -\beta_0 - 1$, i.e.,

$$G(x) = \alpha_0 \, \frac{1+x}{2} - (\beta_0 + 1)\, \frac{1-x}{2}\,.$$

Then we have

$$\begin{aligned} G(x) &= \frac{1}{\pi} \arg\Big(-\mathrm{i}\left[\cos(\ell(x)) + \mathrm{i}\,\sin(\ell(x))\right]\Big) = \frac{1}{\pi}\arg\left[\tan(\ell(x)) - \mathrm{i}\right], \quad \text{where} \\ \ell(x) &= \pi\left(G(x) + \frac{1}{2}\right) = \pi\left[\left(\alpha_0 + \frac{1}{2}\right)\frac{1+x}{2} - \left(\beta_0 + \frac{1}{2}\right)\frac{1-x}{2}\right]. \end{aligned} \tag{2.35}$$

(Remark that $\tan(\ell(\,.\,)) \in \mathbf{C}^\infty[-1, 1]$, since $\ell(x) \in (-\pi/2, \pi/2)$ for all $x \in [-1, 1]$.) The weight function σ_0 belonging to $A = \tan(\ell(\,.\,))\, I + S$ is given by

$$\sigma_0(t) = v^{\alpha_0,\beta_0}(t)\left[\mathrm{e}^{-2}(1-t)^{1-t}(1+t)^{1+t}\right]^{-(\alpha_0+\beta_0+1)/2}\cos(\ell(t))\,. \tag{2.36}$$

As in Theorem 2.13, we introduce two further parameters $k, l \in \{0, 1\}$. Of course, all four parameters α_0, β_0, k, l are uniquely determined by the two values

$$\alpha = \alpha_0 + k \quad \text{and} \quad \beta = \beta_0 + l\,. \tag{2.37}$$

Instead of the operator $A = \tan(\ell(\,.\,))\, I + S$ we consider $A\sigma I$ ($\sigma = v^{k,l}\sigma_0$). This makes it easier to formulate the mapping properties.

Definition 2.16 *Let $\alpha,\beta \in (-1,1)\setminus\{0\}$ and define $\alpha_0,\beta_0 \in (-1,0)$ and $k,l \in \{0,1\}$ by (2.37). The operator $A_{\alpha,\beta}$ is given by*

$$A_{\alpha,\beta} = [\tan(\ell_{\alpha,\beta})\, I + S]\, \sigma_{\alpha,\beta}\, I\,,$$

where $\sigma_{\alpha,\beta} = v^{k,l}\sigma_0$ with σ_0 from (2.36) and $\ell_{\alpha,\beta}(x) = \ell(x)$ with $\ell(x)$ from (2.35).

The weight μ_0 which belongs to the operator $A = \tan(\ell_{\alpha,\beta})\, I + S$ is given by

$$\begin{aligned} \mu_0(t) &= \frac{1}{\sigma_0(t)\,(1+\tan^2(\ell_{\alpha,\beta}(t))} \\ &= v^{-\alpha_0,-\beta_0}(t)\left[\mathrm{e}^{-2}(1-t)^{1-t}(1+t)^{1+t}\right]^{(\alpha_0+\beta_0+1)/2}\cos(\ell_{\alpha,\beta}(t))\,. \end{aligned} \tag{2.38}$$

One can easily show that $\mu_0 = v^{1,1}\sigma_{-\alpha_0-1,-\beta_0-1}$. Consequently, $v^{-k,-l}\mu_0 = \sigma_{-\alpha,-\beta}$ and we obtain

$$A_{-\alpha,-\beta} = -\sigma_{\alpha,\beta}^{-1} B_{k,l}$$

(since $\tan(\ell_{-\alpha,-\beta}) = -\tan(\ell_{\alpha,\beta})$), where $B_{k,l}$ is the operator from Theorem 2.13. To reformulate the mapping properties (2.33) and the assertions of Remark 2.14, we introduce the notation

$$\alpha^{\pm} = \max\{0,\pm\alpha\}\,, \quad \beta^{\pm} = \max\{0,\pm\beta\}$$

and take into account that

$$f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S}) \iff f \in v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\,, \tag{2.39}$$

$$\sigma_{\alpha,\beta} f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S}) \iff f \in v^{\alpha^-,\beta^-}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\,. \tag{2.40}$$

We mention that, writing $f = v^{\alpha^+,\beta^+}g$ and $\sigma_{\alpha,\beta} f = v^{\alpha^+,\beta^+}g$, respectively, (2.39) and (2.40) follow from the equivalence

$$v^{\alpha^+,\beta^+}g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S}) \iff g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S})$$

the proof of which is left to the reader. (For the proof of "$\Rightarrow$" one needs the following consequence of assertion (v) of Corollary 1.18, assertion (vii) of Theorem 1.11, and assertion (i) of Proposition 1.19: If u is a power weight with nonnegative exponents and if $(1\pm x)u(x)g(x)$ is Hölder continuous and vanishes in the zeros of $(1\pm x)u(x)$, then the same is true for $(1\pm x)^{1-\varepsilon}u(x)g(x)$, where $\varepsilon > 0$ is sufficiently small.) Now, (2.33) and Remark 2.14 yield the following.

Proposition 2.17 *$A_{\alpha,\beta}, A_{-\alpha,-\beta}$ can be viewed as operators acting in the following spaces:*

$$A_{\alpha,\beta} \in \mathbb{L}\big(v^{\alpha^-,\beta^-}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}), v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big)\,,\ A_{-\alpha,-\beta} \in \mathbb{L}\big(v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}), v^{\alpha^-,\beta^-}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big).$$

These operators have the following properties:

$$\ker A_{\alpha,\beta} = \Pi_{1-k-l}\,, \quad \ker A_{-\alpha,-\beta} = \Pi_{k+l-1}\,, \quad \textit{where} \quad k = \frac{1+\operatorname{sign}\alpha}{2}\,, \; l = \frac{1+\operatorname{sign}\beta}{2}\,,$$

$$\operatorname{im} A_{\alpha,\beta} = \begin{cases} v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) & \textit{if} \quad k+l \le 1\,, \\ \left\{ f \in v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) : \displaystyle\int_{-1}^{1} f(x)\,\sigma_{-\alpha,-\beta}(x)\,dx = 0 \right\} & \textit{if} \quad k = l = 1\,, \end{cases}$$

$$\operatorname{im} A_{-\alpha,-\beta} = \begin{cases} v^{\alpha^-,\beta^-}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) & \textit{if} \quad k+l \ge 1\,, \\ \left\{ f \in v^{\alpha^-,\beta^-}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) : \displaystyle\int_{-1}^{1} f(x)\,\sigma_{\alpha,\beta}(x)\,dx = 0 \right\} & \textit{if} \quad k = l = 0\,, \end{cases}$$

$$A_{\alpha,\beta}\,A_{-\alpha,-\beta} = -I \;\; \textit{if} \;\; k+l \le 1\,, \quad A_{-\alpha,-\beta}\,A_{\alpha,\beta} = -I \;\; \textit{if} \;\; k+l \ge 1\,.$$

If $k = l = 1$, then $F := I + A_{\alpha,\beta}\,A_{-\alpha,-\beta}$ maps $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ into $\ker A_{-\alpha,-\beta} = \Pi_1$. Hence, F is a linear functional. Moreover, $F(f) - f \in \operatorname{im} A_{\alpha,\beta}$, i.e., $\int_{-1}^{1}[F(f) - f]\,\sigma_{-\alpha,-\beta}\,dx = 0$. Consequently, in every case

$$A_{\alpha,\beta}\,A_{-\alpha,-\beta}\,f = -f + kl\,\frac{\int_{-1}^{1} f(x)\,\sigma_{-\alpha,-\beta}(x)\,dx}{\int_{-1}^{1} \sigma_{-\alpha,-\beta}(x)\,dx}\,, \quad f \in v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\,, \tag{2.41}$$

and, by changing $\alpha \leftrightarrow -\alpha$, $\beta \leftrightarrow -\beta$,

$$A_{-\alpha,-\beta}\,A_{\alpha,\beta}\,f = -f + (1-k)(1-l)\,\frac{\int_{-1}^{1} f(x)\,\sigma_{\alpha,\beta}(x)\,dx}{\int_{-1}^{1} \sigma_{\alpha,\beta}(x)\,dx}\,, \quad f \in v^{\alpha^-,\beta^-}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\,. \tag{2.42}$$

The operator $A_{\alpha,\beta}$ and its above properties play an important role in the following section about regularization of Cauchy singular integral equations. In Chapter 5 we use $A_{\alpha,\beta}$ to construct a numerical method for the approximative solution of such equations. There we will give more details on the action of $A_{\alpha,\beta}$. More precisely, we will describe how the image $A_{\alpha,\beta}\,p$ of a polynomial p can be computed. Here we only mention that $A_{\alpha,\beta}\,p$ is again a polynomial, since Lemma 2.10 applied to $A = \tan(\ell_{\alpha,\beta})\,I + S = A_{\alpha,\beta}\,\sigma_{\alpha,\beta}^{-1}I$ yields

$$A_{\alpha,\beta}(\Pi_n) = A_{\alpha,\beta}\,\sigma_{\alpha,\beta}^{-1}\big(\sigma_0\,v^{k,l}\Pi_n\big) \subseteq \Pi_{n+k+l-1} \quad \text{for all} \;\; n \in \mathbb{N}\,. \tag{2.43}$$

By the way, in view of Proposition 2.5, this property also implies

$$A_{\alpha,\beta} \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}\big)\,, \quad v = u\,v^{-\alpha,-\beta} \tag{2.44}$$

for all $\gamma > 0$, $\delta \in \mathbb{R}$, and all weights u of the type (2.19) which satisfy $\beta \le \tau_0 < 1 + \beta$ and $\alpha \le \tau_{N+1} < 1 + \alpha$. This will be useful in the next section.

2.6 Regularization of Cauchy singular integral equations

Let us consider again an operator $A = aI + SbI$ as in Section 2.3. (We take the same assumptions and notation as in the Sections 2.3 and 2.4.) By a left $\mathbf{C}_u$–regularizer (u a weight of the type (2.19)) of A we mean an operator $\widehat{A}$ such that $\widehat{A}A = I + H$ with some $H \in \mathcal{K}(\mathbf{C}_u)$, where $\mathcal{K}(\mathbf{X})$ denotes the set of all compact linear operators from $\mathbf{X}$ into $\mathbf{X}$. Of course, we have to specify in which sense the equation $\widehat{A}A = I + H$ has to be understood, since A is not a bounded operator in $\mathbf{C}_u$. (In view of Theorem 2.1, we have only $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_{\widetilde{u}})$.) We will do this later. The present section is devoted to the construction of left $\mathbf{C}_u$–regularizers. Before we turn to the details, let us give a preview on the approximation method for the numerical solution of Cauchy singular integral equations which is studied in Chapter 5. This will show us why regularizers are of great importance in our later theory and which properties a regularizer should have to be useful for our purposes.

Let K be some smoothing integral operator (a more precise introduction of K is not necessary at the moment) and let us consider the equation

$$(A + K)f = g\,. \tag{2.45}$$

Depending on the set $\{t_1, \ldots, t_m\} \subseteq [-1,1]$ of singularity points of g, the value of $\varkappa_0$, and expected properties (motivated, for example, by considerations on the underlying physical problem which is modelled by (2.45)) of the solution which is looked for, we fix $k, l \in \{0,1\}$ ($k = 0$ if $\alpha_0 = 0$, $l = 0$ if $\beta_0 = 0$) and look for f in $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$, where $\alpha = \alpha_0 + k$, $\beta = \beta_0 + l$, and $\mathcal{S} = \{t_i\} \cup \{-1,1\}$ (see Remark 2.14 and (2.39)). If we suppose that $g \in v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ and that $\operatorname{im} K \subseteq v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$, then we obtain from Remark 2.14 that a solution $f \in v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$, if it exists, must be of the form

$$f = B_{k,l}(g - Kf) + \sigma\, p = \sigma\big[(aI - SbI)(\mu g - \mu K f) + p\big]\,, \quad p \in \Pi_{-\varkappa}\,. \tag{2.46}$$

The property

$$B_{k,l}(\Pi_n) \subseteq \sigma\Pi_{n+\varkappa} \quad \text{for all} \quad n \in \mathbb{N} \tag{2.47}$$

(this is shown in the proof of Theorem 2.13; see (2.90)) suggests to use so-called spectral methods for the numerical solution of (2.45). This means that we determine approximative solutions $f_n \in \sigma\Pi_{n+\varkappa}$ such that

$$(A + K_n)f_n = g_n\,, \tag{2.48}$$

where $g_n \in \Pi_n$ and $K_n \in \mathbb{L}(\sigma\Pi_{n+\varkappa}, \Pi_n)$ are certain approximations of g and K. Here we remark that (2.48) is an operator equation in the pair $(\sigma\Pi_{n+\varkappa}, \Pi_n)$, since

$$A(\sigma\Pi_{n+\varkappa}) \subseteq A(\sigma_0\Pi_{n+\varkappa_0}) \subseteq \Pi_n \tag{2.49}$$

(see Lemma 2.10). Spectral methods are well studied and in the literature one can find many results about the so-called optimal or almost optimal order of convergence (we will not specify here what this means) of such methods (see, e.g., [CJLM], [JL1], [L1] for investigations of weighted uniform convergence). Of course, spectral methods can only be used if the weight σ is known. But usually the explicite determination of the integral which appears in the definition (2.24) of the part $h(t)$ of $\sigma(t) = v^{\alpha,\beta}(t)h(t)$ is impossible. Thus, in general the values $h(t)$ have to be computed numerically. If we take into account that the practical realization of the method (2.48) requires also the computation of the recurrence coefficients of certain polynomials which are orthogonal with respect to a scalar product in which again the function $h(t)$ appears (see [PS, Sections 9.18–9.23 and 9.39]), it becomes clear that spectral methods are very expensive, except in certain special cases of coefficient functions a and b. For this reason we will only use this approach in case of an operator A of the type $b\,[\tan(\ell_{\alpha,\beta})\,I + S\,]$ (see Section 2.5).

In the literature one can also find, even in the case of operators with piecewise continuous coefficients, convergence and stability results in weighted $\mathbf{L}^2$–spaces for approximation methods of the following type (see [JS4, JR, JRo, JRS, R, W, JW1, JW2]): We look for $f_n \in v^{\rho,\tau}\Pi_n$, where ρ and τ are fixed numbers which are not related to A, such that

$$M_n\big(aI + bS + \widetilde{K}\big)f_n = M_n\,g\,, \tag{2.50}$$

where M_n is a certain projection and $\widetilde{K} = K + SbI - bSI$. (We have used $aI+bS$ instead of A in the above formula, since this is the standard notation in the mentioned literature. In principle this makes no difference, since in Chapter 3 we will see that operators of the type $SbI - bSI$ are smoothing.) One of the advantages is that, if $\rho \neq 0$ and $\tau \neq 0$ are chosen such that $\rho + \tau \in \mathbb{Z}$, it is no problem to determine the images $(aI + bS)f_n$, since then the action of $Sv^{\rho,\tau}I$ on Π is well known (see, e.g., [PS, Section 9.15]). But the error estimates for the method (2.50) are given for the norms of certain weighted Sobolev spaces based on $\mathbf{L}^2$. These estimates do not imply satisfying results about the order of convergence in the norms of weighted spaces of continuous functions. By "satisfying" we mean that the order of convergence of $f_n \to f$ should be not much worse than the order of convergence of $g_n \to g$, at least up to a certain upper bound which depends on the smoothness of the coefficients of A and the kernel of K.

In the present paper we will study compromises between the two above approaches. Namely, as in the spectral methods, we look for approximate solutions of the form $f_n = \widetilde{\sigma}\,p_n$, $p_n \in \Pi$, where $\widetilde{\sigma}$ behaves like $v^{\alpha,\beta}$ (up to positive factors) in the near of -1 and 1. But, as in the second of the above methods, we take a simpler weight $\widetilde{\sigma}$ instead of σ, more precisely, $\widetilde{\sigma} = \sigma_{\alpha,\beta}$ (see Definition 2.16; if $\alpha_0 = 0$, then we take an appropriate other value $\alpha \neq 0$ instead of 0, analogously if $\beta_0 = 0$). Compared with the more simple ansatz $f_n = v^{\alpha,\beta}p_n$ the advantage of the ansatz $f_n = \sigma_{\alpha,\beta}\,p_n$ is that, with the help of an appropriate left regularizer $\widehat{A}$ of A, one can study so-called regularized equations

$$(I + H + \widehat{A}K)f = \widehat{A}g \quad \text{and} \quad (I + H_n + P_n\widehat{A}K_n)f_n = P_n\widehat{A}g_n \tag{2.51}$$

(P_n: some projection onto the ansatz space $\mathbf{X}_n \subseteq \sigma_{\alpha,\beta}\,\Pi$ in which f_n is looked for) instead of the initial equation $(A + K)f = g$ and corresponding approximative equations. More precisely, we will construct $\widehat{A}$ in such a way that the operator H in $\widehat{A}A = I + H$ is a smoothing integral operator the kernel of which can be given explicitely and such that $\widehat{A}g_n$ can be computed for polynomials g_n. We prefer the study of the regularized equations (2.51), since, because of the unboundedness of A in $\mathbf{C}_u$, the initial equation $(A+K)f = g$ cannot be considered in $\mathbf{C}_u$ (which is the space of our interest, since we want to prove weighted uniform convergence), while the equations (2.51) can.

We have written down all these considerations only to motivate the following study of left regularizers of A. The details of the theory of approximation methods for regularized equations are given in Chapters 4 and 5.

We start with an important mapping property of the operator $B_{k,l}$ and A, respectively, which follows from (2.47), (2.49), Proposition 2.5, and (2.11).

Lemma 2.18 *Take k, l, σ, and $B_{k,l}$ as in Theorem 2.13, and let u be a weight of the form (2.19). Further, suppose that*

$$\left\{\begin{array}{ll} \tau_0 \ge -\beta_0 & \text{if } l=0 \\ \tau_0 < -\beta_0 & \text{if } l=1 \end{array}\right\}, \quad \left\{\begin{array}{ll} \tau_{N+1} \ge -\alpha_0 & \text{if } k=0 \\ \tau_{N+1} < -\alpha_0 & \text{if } k=1 \end{array}\right\}. \tag{2.52}$$

Then, for all $\gamma > 0$, $\delta \in \mathbb{R}$,

$$\sigma^{-1}B_{k,l} \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1-s}\big), \quad A\sigma I \in \mathcal{L}\big(\mathbf{C}_v^{\gamma,\delta}, \mathbf{C}_u^{\gamma,\delta-1-s}\big),$$

where $v = v^{\alpha_0+k,\beta_0+l}u$ and $s=1$ if $\tau_0=\beta_0=0$ or $\tau_{N+1}=\alpha_0=0$, $s=0$ otherwise.

Let $g \in \mathbf{C}_u^{\gamma,1+s}$ with u and s as in Lemma 2.18, set $\mathcal{S} = \{x_j\}_{j=0}^{N+1}$, and let the operator K from (2.45) satisfy $K \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}_u^{\gamma,1+s}\big)$ (particularly, $K \in \mathcal{K}(\mathbf{C}_u)$; see Theorem 1.11,(i)). Then, $g \in \mathbf{H}_{\rm loc}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\rm loc}^{k,l}(\mathcal{S})$ (write $v^{-\alpha_0,-\beta_0}g = v^{-\alpha_0-\tau_{N+1},-\beta_0-\tau_0} \cdot ug \cdot v^{\tau_{N+1},\tau_0}/u$ and use assertion (i) of Proposition 1.19) and from Lemma 2.18 and (2.46) we obtain the following: All solutions $f \in \mathbf{C}_u$ of (2.45) for which fu is Hölder continuous (which implies $f \in \mathbf{H}_{\rm loc}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\rm loc}^{k,l}(\mathcal{S})\big)$ belong to $v_0\mathbf{C}_{u_0}^{\min\{\gamma,\eta\},0}$, where

$$u_0 = v^{\alpha_0+k,\beta_0+l}u, \quad v_0 = v^{\alpha_0+k,\beta_0+l}, \tag{2.53}$$

and η is a Hölder exponent of the part h of $\sigma = v_0\,h$. (Remark that $\sigma\mathbf{C}_{u_0}^{\min\{\gamma,\eta\},0} \subseteq v_0\mathbf{C}_{u_0}^{\min\{\gamma,\eta\},0}$ in view of Proposition 1.12.) Moreover, by Lemma 2.18,

$$A \in \mathcal{L}\big(v_0\mathbf{C}_{u_0}^{\gamma,0}, \mathbf{C}_u^{\min\{\gamma,\eta\},-1-s}\big) \quad \text{for all } \gamma > 0$$

(since $\sigma h^{-1}\mathbf{C}_{u_0}^{\min\{\gamma,\eta\},0} \subseteq \sigma\mathbf{C}_{u_0}^{\min\{\gamma,\eta\},0}$).

These facts suggest to redefine the above notion of a left $\mathbf{C}_u$–regularizer as follows.

Definition 2.19 *Let u be a weight of the form (2.19) and define $k, l \in \{0,1\}$ by*

$$\left\{\begin{array}{ll} l=0 & \text{if } \tau_0 \ge -\beta_0 \\ l=1 & \text{if } \tau_0 < -\beta_0 \end{array}\right\}, \quad \left\{\begin{array}{ll} k=0 & \text{if } \tau_{N+1} \ge -\alpha_0 \\ k=1 & \text{if } \tau_{N+1} < -\alpha_0 \end{array}\right\}. \tag{2.54}$$

An operator $\widehat{A}$ is called left $\mathbf{C}_u$–regularizer of A, if there exists a constant $\gamma_0 > 0$ and an operator $H \in \mathcal{L}\big(\mathbf{C}_u, v_0\mathbf{C}_{u_0}^{\gamma_0,0}\big)$ (u_0, v_0 from (2.53)) such that

$$\widehat{A} \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,0}, \mathbf{C}_u\big) \text{ for all } \gamma > 0 \quad \text{and} \quad \widehat{A}Af = f + Hf \text{ for all } f \in \bigcup_{\gamma>0} v_0\mathbf{C}_{u_0}^{\gamma,0}.$$

We remark that, in view of assertion (v) of Corollary 1.18 and assertion (i) of Proposition 1.19, $\bigcup_{\gamma>0} v_0\mathbf{C}_{u_0}^{\gamma,0}$ consists of those elements f of $\mathbf{C}_u$ for which fu is Hölder continuous and vanishes in the zeros of u_0. Furthermore, we mention that we do not need to determine the function $G(x)$ (see (2.23)) to compute the values α_0 and β_0 which are needed in the above definition. Indeed, we have

$$\begin{aligned} \alpha_0 &= G(1) - n_+ \quad \text{with } n_+ \in \mathbb{Z} \text{ such that } \alpha_0 \in (-1,0], \\ -\beta_0 &= G(-1) - n_- \quad \text{with } n_- \in \mathbb{Z} \text{ such that } -\beta_0 \in [0,1), \end{aligned}$$

where $\pi\, G(\pm 1)$ is an argument of $a(\pm 1) - \mathrm{i}\, b(\pm 1) = -\mathrm{i}\left[b(\pm 1) + \mathrm{i}\, a(\pm 1)\right]$, i.e.,

$$G(\pm 1) = -\frac{1}{2} + \frac{1}{\pi} \arg\left[b(\pm 1) + \mathrm{i}\, a(\pm 1)\right] \in -\frac{1}{2} + \left\{ \begin{array}{cc} \frac{1}{2}\,, & b(\pm 1) = 0 \\ \frac{1}{\pi} \arctan \frac{a(\pm 1)}{b(\pm 1)}\,, & b(\pm 1) \neq 0 \end{array} \right\} + \mathbb{Z}\,.$$

Thus, we only need the values of a and b in ± 1 to determine α_0 and β_0,

$$\alpha_0 = \left\{ \begin{array}{cc} 0\,, & b(1) = 0 \\ \frac{1}{\pi} \arctan \frac{a(1)}{b(1)} - \frac{1}{2}\,, & b(1) \neq 0 \end{array} \right\}, \ \beta_0 = \left\{ \begin{array}{cc} 0\,, & b(-1) = 0 \\ -\frac{1}{\pi} \arctan \frac{a(-1)}{b(-1)} - \frac{1}{2}\,, & b(-1) \neq 0 \end{array} \right\}. \quad (2.55)$$

Let u, k, and l as in Definition 2.19. We have already mentioned that every function $f \in \mathbf{C}_u$, for which fu is Hölder continuous, belongs to $\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0} \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$. Particularly, $\bigcup_{\gamma>0} v_0 \mathbf{C}^{\gamma,0}_{u_0} \subseteq \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0} \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$. Now we obtain from Remark 2.14 and Lemma 2.18 that $B_{k,l}$ is a left $\mathbf{C}_u$–regularizer of A (with $H = 0$) if $\varkappa \le 0$ ($\varkappa = \varkappa_0 - k - l$ with $\varkappa_0$ from Theorem 2.11). If $\varkappa > 0$, then we write

$$B_{k,l}\, A = I + H \quad \text{with} \quad H = B_{k,l}\, A - I\,.$$

By Remark 2.14 and (2.33), $H\big(\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0} \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})\big) \subseteq \ker A \cap v^{\alpha_0,\beta_0} \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S})$. Thus,

$$H \in \mathbb{L}\left(\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0} \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\mathcal{S}),\, \sigma \Pi_\varkappa\right)\,. \quad (2.56)$$

Particularly, $H \in \mathbb{L}\big(\bigcup_{\gamma>0} v_0 \mathbf{C}^{\gamma,0}_{u_0}, \sigma\Pi_\varkappa\big)$. To show that H can be extended to a bounded operator from $\mathbf{C}_u$ into $\sigma\Pi_\varkappa$ (where $\sigma\Pi_\varkappa$ can be equipped with any norm, since all norms on this finite-dimensional space are equivalent), we use that, for sufficiently large $p < \infty$, $A \in \mathcal{L}(\mathbf{L}^p_u)$ and $aI - SbI \in \mathcal{L}(\mathbf{L}^p_{u_0})$, where $\mathbf{L}^p_u = \{f : fu \in \mathbf{L}^p(-1,1)\}$, $\|f\|_{\mathbf{L}^p_u} = \|fu\|_p$ (see [MP, Satz II.3.1]). This implies $B_{k,l}A \in \mathcal{L}(\mathbf{L}^p_u)$ and, consequently, $H \in \mathcal{L}(\mathbf{L}^p_u)$. Since Π is dense in $\mathbf{L}^p_u$ (see [MP, Lemma II.3.1]) and, by (2.56), $H(\Pi) \subseteq \sigma\Pi_\varkappa$, we obtain $H \in \mathcal{L}(\mathbf{L}^p_u, \sigma\Pi_\varkappa)$. Obviously, $\mathbf{C}_u$ is continuously embedded into $\mathbf{L}^p_u$ and so the existence of an extension $H \in \mathcal{L}\big(\mathbf{C}_u, \sigma\Pi_\varkappa\big)$ of the operator $B_{k,l}A - I$ is proved. Clearly, $\sigma\Pi_\varkappa$ can be viewed as a subspace of $v_0 \mathbf{C}^{\eta,0}_{u_0}$, where η is a Hölder exponent of the part h of $\sigma = v_0\, h$. Let us summarize.

Proposition 2.20 *Let u be a weight of the form (2.19) and define $k, l \in \{0,1\}$ by (2.54). Then, the operator $B_{k,l}$ from Theorem 2.13 is a left $\mathbf{C}_u$–regularizer of A.*

We have already mentioned that, for our purposes, $B_{k,l}$ is not an appropriate regularizer, since, in general, the computational effort for the determination of the values $\sigma(x)$, $\mu(x)$, and, hence, of the images $B_{k,l}p$ of polynomials p is too high. Moreover, in the case $\varkappa > 0$, we are not able to give a simple representation of the operator H in the sense that H can be written as an integral operator the kernel of which is a simple function with values which can be determined without big effort. However, now we know that an appropriate regularizer should look similar to

$$B_{k,l} = v^{\alpha_0+k,\beta_0+l}\big(aI - h\, S\, h^{-1} b\, I\big)\, \frac{v^{-\alpha_0-k,-\beta_0-l}}{r^2}\, I \qquad \big(r^2 = a^2 + b^2\big)\,. \quad (2.57)$$

Unfortunately, Proposition 2.20 is not useful for us to prove that another operator $\widehat{A}$ is a regularizer, since the second addend of the decomposition $\widehat{A}A = B_{k,l}A + (\widehat{A} - B_{k,l})A$

cannot be considered on $\mathbf{C}_u$ (except we use $\mathbf{L}^p$-theory, which requires that $\widehat{A} - B_{k,l}$ has so good properties, that we cannot expect that $\widehat{A}$ is an appropriate regularizer for our purposes). Of course, one could define the notion of a left regularizer in a sense which is weaker than above (using, e.g., $\mathbf{L}^p$–theory). Then, $B_{k,l}$ could be used to prove that certain other operators are left regularizers. But we will proceed in another way. Namely, we will define an operator $\widehat{A}$ similar to $B_{k,l}$ for which the images of polynomials can be computed and which has the property $\widehat{A}Af = f + Hf$, $f \in \bigcup_{\gamma>0} v_0\mathbf{C}_{u_0}^{\gamma,0}$, with an integral operator H the kernel of which has weak singularities and can be written down explicitely. Then we will show directly that H is compact from $\mathbf{C}_u$ into $v_0\mathbf{C}_{u_0}^{\gamma_0,0}$. This approach is justified by the fact that, anyway, in Chapter 3 we study integral operators with kernels having weak singularities, since later (see Chapter 5) equations of the type (2.45) are considered, where K is an integral operator just of this type.

Let us first search an operator $\widehat{A}$ which looks similar to $B_{k,l}$ and the action of which on polynomials is known. We have already mentioned that the last fact is given for the operators $A_{\alpha,\beta}$ from Section 2.5. If $\alpha_0 \neq 0$ and $\beta_0 \neq 0$, then $\alpha := \alpha_0 + k \in (-1,1) \setminus \{0\}$, $\beta := \beta_0 + l \in (-1,1) \setminus \{0\}$, and in the near of ± 1 the weight $\sigma_{-\alpha,-\beta}$ (see Definition 2.16) behaves like the weight $v^{-\alpha_0-k,-\beta_0-l}$ on the right hand side of (2.57) (up to positive factors). This suggests to use $A_{-\alpha,-\beta}$ in the construction of $\widehat{A}$. Taking into account that the difference between $h\,S\,h^{-1}b\,I$ and bS is a smoothing integral operator (here we refer to later considerations; see Chapter 3) and that, in view of (2.57), the multiplication operator $(a/r^2)\,I$ should appear in $\widehat{A}$, we can hope that

$$\widehat{A} := \frac{1}{r^2\,\sigma_{-\alpha,-\beta}}\big(aI - bS\big)\,\sigma_{-\alpha,-\beta}\,I = \frac{1}{r^2}\big[a - b\tan(\ell_{\alpha,\beta})\big]I - \frac{b}{r^2\,\sigma_{-\alpha,-\beta}}\,A_{-\alpha,-\beta} \tag{2.58}$$

is an appropriate left $\mathbf{C}_u$–regularizer of A. (Remark that $\tan(\ell_{\alpha,\beta}) = -\tan(\ell_{-\alpha,-\beta})$.) To check this, we first have to compute $\widehat{A}A$. For this aim, let us forget for a moment the weight u and consider (2.58) for arbitrary $\alpha, \beta \in (-1,1) \setminus \{0\}$. To reduce the length of the formulas, we introduce the notation

$$\widetilde{a} := a - b\tan(\ell_{\alpha,\beta})\,, \quad \widetilde{k} := \left\{ \begin{array}{ll} 0\,, & \alpha > 0 \\ 1\,, & \alpha < 0 \end{array} \right\}, \quad \widetilde{l} := \left\{ \begin{array}{ll} 0\,, & \beta > 0 \\ 1\,, & \beta < 0 \end{array} \right\}. \tag{2.59}$$

Then, by (2.42), we obtain the following identity, which holds on $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$:

$$\begin{aligned}
r^2\widehat{A}A
&= \left(\widetilde{a}\,I - \frac{b}{\sigma_{-\alpha,-\beta}}\,A_{-\alpha,-\beta}\right)\left(\widetilde{a}\,I + A_{\alpha,\beta}\,\frac{b}{\sigma_{\alpha,\beta}}\,I\right) \\
&= \left(\widetilde{a}^2 + \frac{b^2}{\sigma_{\alpha,\beta}\,\sigma_{-\alpha,-\beta}}\right)I + \widetilde{a}\,A_{\alpha,\beta}\,\frac{b}{\sigma_{\alpha,\beta}}\,I - \frac{b}{\sigma_{-\alpha,-\beta}}\,A_{-\alpha,-\beta}\,\widetilde{a}\,I - \frac{\widetilde{k}\,\widetilde{l}\,b}{\sigma_{-\alpha,-\beta}}\,\frac{\int_{-1}^1 \,.\, b\,dx}{\int_{-1}^1 \sigma_{\alpha,\beta}\,dx} \\
&= \left(\widetilde{a}^2 + \frac{b^2}{\sigma_{\alpha,\beta}\,\sigma_{-\alpha,-\beta}} + 2\,\widetilde{a}\,b\tan(\ell_{\alpha,\beta})\right)I + \widetilde{a}\,SbI - \frac{b}{\sigma_{-\alpha,-\beta}}\,S\,\widetilde{a}\,\sigma_{-\alpha,-\beta}\,I \\
&\qquad - \frac{\widetilde{k}\,\widetilde{l}\,b}{\sigma_{-\alpha,-\beta}}\,\frac{\int_{-1}^1 \,.\, b\,dx}{\int_{-1}^1 \sigma_{\alpha,\beta}\,dx} \\
&= r^2 I + \widetilde{a}\,SbI - \frac{b}{\sigma_{-\alpha,-\beta}}\,S\,\widetilde{a}\,\sigma_{-\alpha,-\beta}\,I - \frac{\widetilde{k}\,\widetilde{l}\,b}{\sigma_{-\alpha,-\beta}}\,\frac{\int_{-1}^1 \,.\, b\,dx}{\int_{-1}^1 \sigma_{\alpha,\beta}\,dx}\,,
\end{aligned} \tag{2.60}$$

where $\int_{-1}^{1} \,.\, b\,dx$ denotes the operator $f \to \int_{-1}^{1} f\,b\,dx$. We remark that the last equality follows from $\sigma_{-\alpha,-\beta}^{-1} = \sigma_{\alpha,\beta}\big[1+\tan^2(\ell_{\alpha,\beta})\big]$ (see (2.38)) and that $\widehat{A}A$ is well defined on $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ since $A_{\alpha,\beta}\,\sigma_{\alpha,\beta}^{-1}I$, $\sigma_{-\alpha,-\beta}^{-1}A_{-\alpha,-\beta} \in \mathbb{L}\big(v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big)$ (see Proposition 2.17) imply

$$A, \widehat{A} \in \mathbb{L}\big(v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big)\,. \tag{2.61}$$

Now we ask the following question: Is it possible to define α and β in dependence of a given weight u of the type (2.19) in such a way that the above operator $\widehat{A}$ is a left $\mathbf{C}_u$–regularizer of A? First we remark that, by (2.58) and (2.44),

$$\widehat{A} \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,0}, \mathbf{C}_u\big) \quad \text{for all } \gamma > 0 \tag{2.62}$$

if α and β satisfy $\tau_{N+1}-1 < -\alpha \le \tau_{N+1}$ and $\tau_0 - 1 < -\beta \le \tau_0$. This shows that in the case $\alpha_0 \neq 0$ the choice $\alpha = \alpha_0 + k$ with k from (2.54) is admissible (analogously, $\beta = \beta_0 + l$ if $\beta_0 \neq 0$). Let us mention that, in this case, the definitions (2.35) and (2.55) of $\ell = \ell_{\alpha,\beta}$ and α_0, β_0 imply

$$\tan\big(\ell_{\alpha,\beta}(1)\big) = \frac{a(1)}{b(1)} \qquad \left(\tan\big(\ell_{\alpha,\beta}(-1)\big) = \frac{a(-1)}{b(-1)}\right),$$

i.e., $\widetilde{a}(1) = 0$ ($\widetilde{a}(-1) = 0$). We will see that this property of $\widetilde{a}$ or, more precisely, the property

$$\big(\widetilde{a}\,b\big)(-1) = \big(\widetilde{a}\,b\big)(1) = 0 \tag{2.63}$$

is important if one wants to prove that the part

$$\frac{\widetilde{a}}{r^2}\,SbI - \frac{b}{r^2\,\sigma_{-\alpha,-\beta}}\,S\,\widetilde{a}\,\sigma_{-\alpha,-\beta}\,I$$

of $\widehat{A}A$ (see (2.60)) is a bounded operator from $\mathbf{C}_u$ into $v_0\mathbf{C}_{u_0}^{\gamma_0,0}$ (comp. Definition 2.19). If $\alpha_0 = 0$, i.e., $b(1) = 0$, then we can take an arbitrary $\alpha \neq 0$ with $\tau_{N+1} - 1 < -\alpha \le \tau_{N+1}$ (analogously, $\beta \neq 0$ with $\tau_0 - 1 < -\beta \le \tau_0$ if $\beta_0 = 0$) to ensure that (2.63) and (2.62) hold true. From the inequalities $\tau_{N+1} - 1 < -\alpha \le \tau_{N+1}$ and $\tau_0 - 1 < -\beta \le \tau_0$ it follows also that every $f \in \mathbf{C}_u$ with Hölder continuous fu belongs to $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$, particularly,

$$\bigcup_{\gamma>0} v_0\mathbf{C}_{u_0}^{\gamma,0} \subseteq v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$$

(comp. Definition 2.19 and remember that (2.60) holds on $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$). Now we know in every case how α and β must be chosen to have some hope that $\widehat{A}$ defined by (2.58) is a left $\mathbf{C}_u$–regularizer of A. It turns out that this is really the case just for these choices of α and β (only in the case $\alpha_0 = 0$ we have to take $\tau_{N+1} > 0$ and $\alpha > -\tau_{N+1}$; analogously $\tau_0 > 0$ and $\beta > -\tau_0$ if $\beta_0 = 0$). The proof is given later. Let us sum up with the following theorem in which also the notation is repeated (for the convenience of the reader) and which contains a more detailed description of the mapping properties of H.

Theorem 2.21 *Let u be a weight of the form (2.19), where $\tau_0 > 0$ if $b(-1) = 0$, $\tau_{N+1} > 0$ if $b(1) = 0$, and define k, l by (2.54). Then, for*

$$\left\{\begin{array}{cc} \alpha = \alpha_0 + k\,, & b(1) \neq 0 \\ \alpha \in (-\tau_{N+1}, 1-\tau_{N+1}) \setminus \{0\}\,, & b(1) = 0 \end{array}\right\}, \quad \left\{\begin{array}{cc} \beta = \beta_0 + l\,, & b(-1) \neq 0 \\ \beta \in (-\tau_0, 1-\tau_0) \setminus \{0\}\,, & b(-1) = 0 \end{array}\right\}$$

(with α_0 and β_0 from (2.55)), the operator

$$\widehat{A} = \frac{1}{r^2\,\sigma_{-\alpha,-\beta}}\,\big(aI - bS\big)\,\sigma_{-\alpha,-\beta}\,I$$

(comp. (2.58)) is a left $\mathbf{C}_u$–regularizer of $A = aI + SbI$. Here, $r^2 = a^2 + b^2$ and

$$\sigma_{-\alpha,-\beta}(t) = v^{-\alpha,-\beta}(t)\left[\mathrm{e}^{-2}(1-t)^{1-t}(1+t)^{1+t}\right]^{\frac{\alpha+\beta-(\operatorname{sign}\alpha+\operatorname{sign}\beta)/2}{2}}\cos(\ell_{\alpha,\beta}(t))\,,\quad \textit{where}$$

$$\ell_{\alpha,\beta}(t) = \pi\left[\left(\alpha - \frac{\operatorname{sign}\alpha}{2}\right)\frac{1+t}{2} - \left(\beta - \frac{\operatorname{sign}\beta}{2}\right)\frac{1-t}{2}\right].$$

Moreover, $\widehat{A}Af = f + Hf$ for all $f \in v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}\Big(\{x_i\}_{i=0}^{N+1}\Big)$ (particularly, for all $f \in \mathbf{C}_u$ with Hölder continuous fu), where $(Hf)(x)$, $x \in (-1,1)\setminus\{x_i\}_{i=1}^{N}$, is given by

$$\frac{1}{\pi\,\sigma_{-\alpha,-\beta}(x)\,r^2(x)}\int_{-1}^{1}\left[\frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\,b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\,b(x)}{t-x} - \frac{\pi\,\widetilde{k}\,\widetilde{l}\,b(x)\,b(t)}{\int_{-1}^{1}\sigma_{\alpha,\beta}(\tau)\,d\tau}\right] f(t)\,dt$$

($\widetilde{a}$, $\widetilde{k}$, and $\widetilde{l}$ are defined in (2.59). $\sigma_{\alpha,\beta}$ can be obtained by replacing $-\alpha \leftrightarrow \alpha$ and $-\beta \leftrightarrow \beta$ in the above definition of $\sigma_{-\alpha,-\beta}$.) The operator H has the property

$$\sigma_{-\alpha,-\beta}\,r^2\,H \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(\alpha,\beta)}\big) \tag{2.64}$$

if the following assumptions on γ, δ, a, and b are satisfied:

$$\begin{cases} \gamma \in (0,2)\setminus\{1\}\,,\ \delta \le -1\,,\ a^{([\gamma])}, b^{([\gamma])} \in \mathbf{C}^{\gamma-[\gamma],\delta+1} & \textit{if}\quad b(1)b(-1) \ne 0\,, \\ \gamma \in (0,1)\,,\ \delta \le -1\,,\ a \in \mathbf{C}^{\gamma,\delta+1},\ b \in \mathbf{C}^{2\gamma,\delta+1} & \textit{if}\quad b(1)b(-1) = 0\,, \end{cases}$$

($[\gamma]$ denotes the integer part of γ) and, in the case $b(1)b(-1) = 0$,

$$\begin{cases} \gamma \le 2\tau_{N+1}\,,\ \gamma < 2(1-\alpha-\tau_{N+1})\,, & b(1) = 0,\ b(-1) \ne 0, \\ \gamma \le 2\tau_0\,,\ \gamma < 2(1-\beta-\tau_0)\,, & b(1) \ne 0,\ b(-1) = 0, \\ \gamma \le 2\min\{\tau_0,\tau_{N+1}\}\,,\ \gamma < 2\min\{1-\alpha-\tau_{N+1},\, 1-\beta-\tau_0\}\,, & b(1) = 0,\ b(-1) = 0. \end{cases}$$

2.7 Proofs

We will use the following convention for integrals of the form

$$\int_I \frac{g(t)}{t-x}\,dt\,. \tag{2.65}$$

If x is an inner point of I and g does not depend on x, then we mean the Cauchy principle value. In all other cases, (2.65) is a Lebesgue integral.

2.7.1 Proof of Theorem 2.1

Lemma 2.22 *Let $\beta \in [0,1)$ and $y \in [-1,1]$ be fixed. Then,*

$$\int_{-1}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| |t-y|^{-\beta}\, dt \le c\, |x-y|^{-\beta}\, \|f\|_0 \tag{2.66}$$

for all $f \in \mathbf{C}^0$ and all $x \in [-1,1]$ ($x \neq y$ if $\beta > 0$), where $c \neq c(f,x)$.

Proof. If $\beta = 0$, then we can use Theorem 1.13 to obtain the assertion. Indeed,

$$\begin{aligned} \int_{-1}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| dt \;&\le\; \int_{-1}^{1} \frac{\omega(f,|t-x|)}{|t-x|}\, dt = \int_{-1-x}^{1-x} \frac{\omega(f,|h|)}{|h|}\, dh \\ &\le\; 2\int_0^2 \frac{\omega(f,h)}{h}\, dh \le c\, \|f\|_0\,. \end{aligned} \tag{2.67}$$

Now, let $\beta \in (0,1)$. If we succeed in proving the estimate

$$\int_{-1}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| |t-y|^{-\beta}\, dt \le c\, |x-y|^{-\beta} \left(\|f\| + \int_{-1}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| dt \right), \tag{2.68}$$

then, in view of (2.67), (2.66) follows. To verify (2.68), we first consider the case $x > y$. In this case, the left hand side of (2.68) can be estimated by

$$\begin{aligned} 2\|f\| &\left[\int_{-1}^{y-\frac{x-y}{2}(y+1)} \frac{(y-t)^{-\beta}}{x-t}\, dt + \int_{y-\frac{x-y}{2}(y+1)}^{\frac{x+y}{2}} \frac{|t-y|^{-\beta}}{x-t}\, dt \right] + \\ &+ \int_{\frac{x+y}{2}}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| (t-y)^{-\beta} dt \;=: 2\|f\|\,[I_1 + I_2] + I_3\,. \end{aligned}$$

If $y = -1$, then I_1 vanishes. Otherwise we use that $x - t \ge y - t$ in the first integral,

$$I_1 \le \int_{-1}^{y-\frac{x-y}{2}(y+1)} (y-t)^{-\beta-1} dt \le c\, (x-y)^{-\beta}\,.$$

In integral I_2 we have $x - t \ge (x-y)/2$. Consequently,

$$I_2 \le 2(x-y)^{-1} \int_{y-\frac{x-y}{2}(y+1)}^{\frac{x+y}{2}} |t-y|^{-\beta}\, dt \le c\, (x-y)^{-\beta}\,.$$

For $t \ge (x+y)/2$, $(t-y)^{-\beta}$ can be estimated by $2^{\beta}(x-y)^{-\beta}$ and we obtain

$$I_3 \le c\, (x-y)^{-\beta} \int_{-1}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| dt\,.$$

Thus, (2.68) is proved in case $x > y$. If $x < y$, then one can proceed in a similar way or one uses the substitution $\tau = -t$ which makes it possible to apply what we have already proved (with $-x$ and $-y$ instead of x and y). ∎

Lemma 2.23 *Let $\beta \in [0,1)$, $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, and $\sigma \in \{0,1\}$ be fixed, where $\sigma = 0$ if $\beta = 0$. Then, for all $f \in \mathbf{C}^0$,*

$$\left| \int_{-1}^{1} \frac{f(t)}{t-x} \frac{\left[\mathrm{sign}(t-y)\right]^\sigma}{|t-y|^\beta} \, dt \right| \le c \, \frac{\|f\|_0}{|x-y|^\beta} \,, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{2.69}$$

($x \neq y$ if $\beta > 0$), where $c \neq c(f,x)$.

Proof. The integral on the left hand side of (2.69) can be written as

$$\int_{-1}^{1} \frac{f(t)-f(x)}{t-x} \frac{\left[\mathrm{sign}(t-y)\right]^\sigma}{|t-y|^\beta} \, dt + f(x) \int_{-1}^{1} \frac{\left[\mathrm{sign}(t-y)\right]^\sigma}{|t-y|^\beta} \, \frac{dt}{t-x} \,. \tag{2.70}$$

In view of (2.66), the absolute value of the first addend can be estimated by $c\,|x-y|^{-\beta}\|f\|_0$. Moreover, in the case $\beta = \sigma = 0$, the second addend equals

$$f(x) \int_{-1}^{1} \frac{dt}{t-x} = f(x) \ln \frac{1-x}{1+x}$$

and (2.69) is proved. It remains to estimate the last integral in (2.70) for $\beta > 0$ and $x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{y\}$. This integral can be decomposed into

$$\int_{-1}^{1} \left[\frac{\left[\mathrm{sign}(t-y)\right]^\sigma}{|t-y|^\beta} - \frac{\left[\mathrm{sign}(x-y)\right]^\sigma}{|x-y|^\beta} \right] \frac{dt}{t-x} + \frac{\left[\mathrm{sign}(x-y)\right]^\sigma}{|x-y|^\beta} \ln \frac{1-x}{1+x} \,.$$

Together with the substitution $t - y = \tau(x-y)$ we obtain

$$\int_{-1}^{1} \frac{\left[\mathrm{sign}(t-y)\right]^\sigma}{|t-y|^\beta} \, \frac{dt}{t-x} = \frac{\left[\mathrm{sign}(x-y)\right]^\sigma}{|x-y|^\beta} \left[\int_{-\frac{1+y}{x-y}}^{\frac{1-y}{x-y}} \frac{(\mathrm{sign}\,\tau)^\sigma |\tau|^{-\beta} - 1}{\tau - 1} \, d\tau + \ln \frac{1-x}{1+x} \right].$$

If $|x-y| \ge 1/4$, then

$$\left| \int_{-\frac{1+y}{x-y}}^{\frac{1-y}{x-y}} \frac{(\mathrm{sign}\,\tau)^\sigma |\tau|^{-\beta} - 1}{\tau - 1} \, d\tau \right| \le \int_{-6}^{6} \left| \frac{(\mathrm{sign}\,\tau)^\sigma |\tau|^{-\beta} - 1}{\tau - 1} \right| d\tau = c < \infty \,. \tag{2.71}$$

Now, let $|x-y| < 1/4$. Without loss of generality we may suppose $x > y$. (If $x < y$, then we only have to change the roles of $1+y$ and $1-y$ in the following considerations.) The left hand side of (2.71) can be estimated by

$$\int_{-2}^{2} \left| \frac{(\mathrm{sign}\,\tau)^\sigma |\tau|^{-\beta} - 1}{\tau - 1} \right| d\tau + \left(\int_{-\infty}^{-2} + \int_{2}^{\infty} \right) \frac{|\tau|^{-\beta}}{|\tau - 1|} \, d\tau + \left| \left(\int_{-\frac{1+y}{x-y}}^{-2} + \int_{2}^{\frac{1-y}{x-y}} \right) \frac{d\tau}{\tau - 1} \right| .$$

The first two integrals are (finite) constants (use $|\tau - 1| \sim |\tau|$, $|\tau| \ge 2$). The last integral can be computed explicitly,

$$\left(\int_{-\frac{1+y}{x-y}}^{-2} + \int_{2}^{\frac{1-y}{x-y}} \right) \frac{d\tau}{\tau - 1} = \ln \frac{3(1-x)}{1+x} \,.$$

Hence, its absolute value is uniformly bounded for $|x| \le 1/2$. ∎

Lemma 2.24 *Let $\beta \in [0,1)$, $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ be fixed. Then, for all $f \in \mathbf{C}^0$,*

$$\left| \int_\xi^1 \frac{f(t)}{t-x} \, \frac{dt}{|t-y|^\beta} \right| \le c \, \frac{\|f\|_0}{|x-y|^\beta} \, \ln \frac{\mathrm{e}}{|x-\xi|^\eta} \,, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{2.72}$$

($x \neq y$ if $\beta > 0$, $x \neq \xi$ if $\eta > 0$), where

$$\eta = \begin{cases} 0\,, & f(\xi) = 0 \ \text{ or } \ y = \xi \text{ and } \beta > 0 \\ 1\,, & \text{otherwise} \end{cases} \tag{2.73}$$

and $c \neq c(f,x)$.

Proof. Let $y \neq \xi$ or $\beta = 0$. Denote by χ_ξ the characteristic function of the interval $[\xi, 1]$ and define

$$g(t) := \chi_\xi(t) \left(f(t) - f(\xi) \, \frac{|t-y|^\beta}{|\xi-y|^\beta} \right).$$

Then, $g \in \mathbf{C}^0$ and $\|g\|_0 \le c \, \|f\|_0$. (Apply (2.3) with $[c,d] = [\xi, 1]$ and with g instead of f.) Moreover,

$$\int_\xi^1 \frac{f(t)}{t-x} \, \frac{dt}{|t-y|^\beta} = \int_{-1}^1 \frac{g(t)}{t-x} \, \frac{dt}{|t-y|^\beta} + \frac{f(\xi)}{|\xi-y|^\beta} \int_\xi^1 \frac{dt}{t-x} \,. \tag{2.74}$$

In view of Lemma 2.23, we may estimate

$$\left| \int_{-1}^1 \frac{g(t)}{t-x} \, \frac{dt}{|t-y|^\beta} \right| \le c \, \frac{\|f\|_0}{|x-y|^\beta} \,, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{2.75}$$

If $f(\xi) = 0$, then this is the assertion. If $f(\xi) \neq 0$, then $\eta = 1$ and (2.72) follows from (2.74) and (2.75), since

$$\int_\xi^1 \frac{dt}{t-x} = \ln \frac{1-x}{|x-\xi|} \,.$$

Now, let $y = \xi$ and $\beta > 0$. Then we write

$$2 \int_y^1 \frac{f(t)}{t-x} \, \frac{dt}{|t-y|^\beta} = \int_{-1}^1 \frac{f(t)}{t-x} \, \frac{dt}{|t-y|^\beta} + \int_{-1}^1 \frac{f(t)}{t-x} \, \frac{\operatorname{sign}(t-y)}{|t-y|^\beta} \, dt$$

and the assertion follows from Lemma 2.23. ■

Corollary 2.25 *Let $\beta \in [0,1)$, $y \in [-1,1]$, $\xi \in [-1,1]$ be fixed. Then, for all $f \in \mathbf{C}^0[-2,2]$ (see (2.2) for the definition of $\mathbf{C}^0[-2,2]$),*

$$\left| \int_\xi^2 \frac{f(t)}{t-x} \, \frac{dt}{|t-y|^\beta} \right| \le c \, \frac{\|f\|_{\mathbf{C}^0[-2,2]}}{|x-y|^\beta} \, \ln \frac{2\,\mathrm{e}}{|x-\xi|^\eta} \,, \quad x \in [-1,1] \tag{2.76}$$

($x \neq y$ if $\beta > 0$, $x \neq \xi$ if $\eta > 0$), where η is defined in (2.73) and $c \neq c(f,x)$.

Proof. The substitution $t = 2\tau$ shows that

$$\int_\xi^2 \frac{f(t)}{t-x} \frac{dt}{|t-y|^\beta} = \frac{1}{2^\beta} \int_{\xi/2}^1 \frac{f(2\tau)}{\tau-(x/2)} \frac{d\tau}{|\tau-(y/2)|^\beta} .$$

In view of Theorem 1.13, the function $g(\tau) = f(2\tau)$ belongs to $\mathbf{C}^0$, where $\|g\|_0 \le c\,\|f\|_{\mathbf{C}^0[-2,2]}$ (since $\omega(g,h) = \omega_{[-2,2]}(f,2h)$). Now, Lemma 2.24 yields the assertion. ■

Proof of Theorem 2.1. Let $f \in \mathbf{C}^0_u$. In view of Theorem 1.13, $g := fu$ can be viewed as an element of $\mathbf{C}^0[-2,2]$ (set $g(x) = g(1)$ for $x > 1$ and $g(x) = g(-1)$ for $x < -1$), where $\|g\|_{\mathbf{C}^0[-2,2]} \le c\,\|f\|_{u,0}$. (Use that $\omega_{[-2,2]}(g,h) = \omega(g,h)$.) If we write $wf = v^{-1}g$ and take into account that, obviously,

$$\|av^{-1}I\|_{\mathbf{C}\to\mathbf{B}_{v[\mathcal{M}]}} \le c\,\|a\|\,, \quad \|b_1 I\|_{\mathbf{B}_{v[\mathcal{M}]}\to\mathbf{B}_{v[\mathcal{M}]}} \le \|b_1\|\,,$$

then it is clear that it remains to prove

$$\|Sb_2v^{-1}g\|_{v[\mathcal{M}]} \le c\,\|g\|_{\mathbf{C}^0[-2,2]} \max\left\{\big\|b_2^{(0)}\big\|_0, \big\|b_2^{(1)}\big\|_0, \ldots, \big\|b_2^{(R)}\big\|_0\right\}. \tag{2.77}$$

For this aim, we define functions h_k, $k = 0,\ldots,M$, as follows. If $M = 0$, then $h_0 \equiv 1$. If $M > 0$, then $h_0 \equiv 0$ and

$$h_k(t) = \frac{\prod\limits_{1=j\ne k}^{M} |t-y_j|}{\prod\limits_{1=j\ne 1}^{M} |t-y_j|^{\beta_j+1} + \ldots + \prod\limits_{1=j\ne M}^{M} |t-y_j|^{\beta_j+1}}\,, \quad k = 1,\ldots,M\,.$$

If we set $\beta_0 = y_0 = 0$, then we may write

$$v^{-1}(t) = \sum_{k=0}^{M} \frac{h_k(t)}{|t-y_k|^{\beta_k}}\,. \tag{2.78}$$

Moreover, we extend all function $b_2^{(j)}$ onto $[-2,2]$ such that

$$\big\|b_2^{(j)}\big\|_{\mathbf{C}^0[-2,2]} \le c\,\big\|b_2^{(j)}\big\|_0\,, \quad j = 0,\ldots,R\,. \tag{2.79}$$

(Clearly, this can be done in the same way as above for $g = fu$.) Let us denote by χ_j and $\chi_{j,\infty}$ the characteristic functions of the interval $[\xi_j,\xi_{j+1})$ and $[\xi_j,\infty)$, respectively. Then the function

$$\sum_{j=0}^{R} b_2^{(j)}\chi_j = \sum_{j=0}^{R}\left(b_2^{(j)}\chi_{j,\infty} - b_2^{(j)}\chi_{j+1,\infty}\right)$$

equals b_2 on $(-1,1)\setminus\{\xi_l\}_{l=1}^R$ and vanishes on $[1,2]$. Together with (2.78) we obtain

$$\begin{aligned}\pi\big(Sb_2v^{-1}g\big)(x) &= \int_{-1}^{2} \frac{v^{-1}(t)\,g(t)}{t-x} \sum_{j=0}^{R}\left(b_2^{(j)}(t)\,\chi_{j,\infty}(t) - b_2^{(j)}(t)\,\chi_{j+1,\infty}(t)\right)dt \\ &= \sum_{k=0}^{M}\sum_{j=0}^{R}\sum_{l=j}^{j+1}(-1)^{l-j}\int_{\xi_l}^{2} \frac{g(t)\,b_2^{(j)}(t)\,h_k(t)}{t-x} \frac{dt}{|t-y_k|^{\beta_k}}\,.\end{aligned}$$

The function h_k is Lipschitz continuous on $[-2,2]$ (particularly, $h_k \in \mathbf{C}^0[-2,2]$) and vanishes in all y_i with $i \in \{1,\ldots,M\} \setminus \{k\}$. Together with

$$\|f_1 f_2\|_{\mathbf{C}^0[-2,2]} \le \|f_1\|_{\mathbf{C}^0[-2,2]} \|f_2\|_{\mathbf{C}^0[-2,2]}$$

(since $\omega_{[-2,2]}(f_1 f_2, h) \le \|f_1\|_{\mathbf{C}[-2,2]} \omega_{[-2,2]}(f_2,h) + \|f_2\|_{\mathbf{C}[-2,2]} \omega_{[-2,2]}(f_1,h)$), (2.79), and $g(x_i) = 0$, $i = 1,\ldots,N$ (see Theorem 1.13) we conclude

$$\begin{aligned} &\|g\, b_2^{(j)}\, h_k\|_{\mathbf{C}^0[-2,2]} \le c\, \|g\|_{\mathbf{C}^0[-2,2]} \|b_2^{(j)}\|_0 \,, \\ &\big(g\, b_2^{(j)}\, h_k\big)(x_i) = 0\,, \quad i \in \{1,\ldots,N\}\,, \\ &\big(g\, b_2^{(j)}\, h_k\big)(y_i) = 0\,, \quad i \in \{1,\ldots,M\} \setminus \{k\}\,. \end{aligned}$$

Now, if $\xi_l \in \{y_1,\ldots,y_M\}$, then in both cases $\xi_l = y_k$ and $\xi_l = y_i$, $i \ne k$, Corollary 2.25 shows that

$$\left| \int_{\xi_l}^2 \frac{g(t)\, b_2^{(j)}(t)\, h_k(t)}{t-x} \frac{dt}{|t-y_k|^{\beta_k}} \right| \le c\, \frac{\|g\|_{\mathbf{C}^0[-2,2]} \|b_2^{(j)}\|_0}{|x-y_k|^{\beta_k}}\,. \tag{2.80}$$

The same holds if $\xi_l \in \{x_1,\ldots,x_N\}$. If $\xi_l \notin \{x_1,\ldots,x_N\} \cup \{y_1,\ldots,x_M\}$, then we have to modify the right hand side of (2.80) by the factor $\ln \dfrac{2\,\mathrm{e}}{|x-\xi_l|}$. Thus, (2.77) is proved. ■

2.7.2 Proofs of Theorem 2.11 and Proposition 2.9

The following properties of $A = aI + SbI$ and $B = \sigma_0(aI - SbI)\mu_0 I$ can be found in [PS, Theorem 9.9 and Remark 9.10].

Lemma 2.26 *$A(\sigma_0\Pi_n) \subseteq \Pi_{n-\varkappa_0}$, $B(\Pi_n) \subseteq \sigma_0\Pi_{n+\varkappa_0}$, and A, B considered as operators $A \in \mathbb{L}(\sigma_0\Pi, \Pi)$ and $B \in \mathbb{L}(\Pi, \sigma_0\Pi)$ have the following properties:*

$$\ker A = \sigma_0\Pi_{\varkappa_0}\,, \quad \ker B = \Pi_{-\varkappa_0}\,, \quad BA = I \ \text{ if } \varkappa_0 \le 0\,, \quad AB = I \ \text{ if } \varkappa_0 \ge 0\,. \tag{2.81}$$

Proof of Theorem 2.11. In view of Proposition 2.5, we have

$$\sigma_0^{-1} B \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_{u_0}^{\gamma,\delta-1},\big)\,, \quad u_0 = v^{\alpha_0,\beta_0} u \tag{2.82}$$

for all $\gamma > 0$, $\delta \in \mathbb{R}$, and all weights u of the form (2.19) which satisfy $\tau_0 > -\beta_0$ and $\tau_{m+1} > -\alpha_0$. We have already shown that the corresponding property (2.21) of $A\sigma_0 I$ implies $A \in \mathbb{L}\big(\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big)$. In a similar way we can prove

$$B \in \mathbb{L}\big(\mathbf{H}\big)\,, \quad \mathbf{H} = \mathbf{H}_{\mathrm{loc}}(\mathcal{S})\,.$$

Now we show that the assertions (2.81) of Lemma 2.26 remain true if A and B are considered as operators in $\mathbf{H}$. To prove $\ker A = \sigma_0\Pi_{\varkappa_0}$, we only have to show $\ker A \subseteq \sigma_0\Pi_{\varkappa_0}$, since the reverse inclusion follows from Lemma 2.26. Let $f \in \mathbf{H}$ such that $Af = 0$. Then there exists a weight u of the above form and some $\gamma > 0$ such that

$$\sigma_0^{-1} f \in \mathbf{C}_{u_0}^{\gamma,0}$$

(see the proof of (2.22)). Without loss of generality we may assume that

$$\lim_{n\to\infty} n^\gamma E_n^{u_0}(\sigma_0^{-1}f) = 0\,. \tag{2.83}$$

(Replace γ by some $\widetilde{\gamma} < \gamma$ if this is not satisfied.) In view of assertion (iii) of Theorem 1.11, (2.83) means that $\sigma_0^{-1}f$ can be approximated by polynomials p_n in the norm of $\mathbf{C}_{u_0}^{\gamma,0}$. Together with (2.21) we obtain

$$A\sigma_0\, p_n \to Af = 0 \;\text{ in }\; \mathbf{C}_u^{\gamma,-1} \quad\text{and}\quad \sigma_0\, p_n \to f \;\text{ in }\; \mathbf{C}_u\,. \tag{2.84}$$

The first part of (2.84) and (2.82) imply $\sigma_0^{-1}BA\sigma_0 p_n \to 0$ in $\mathbf{C}_{u_0}$. Hence,

$$f_n := \sigma_0\, p_n - BA\sigma_0\, p_n \to f \;\text{ in }\; \mathbf{C}_u\,.$$

By Lemma 2.26, $A(I - BA) = 0$ on $\sigma_0\Pi$. Consequently, all f_n belong to the kernel of $A \in \mathbb{L}(\sigma_0\Pi, \Pi)$, i.e., $f_n \in \sigma_0\Pi_{\varkappa_0}$ for all n. Clearly, $\sigma_0\Pi_{\varkappa_0}$ is a finite-dimensional and, hence, closed subspace of $\mathbf{C}_u$. Thus, $f = \mathbf{C}_u$–$\lim f_n \in \sigma_0\Pi_{\varkappa_0}$ and $\ker A = \sigma_0\Pi_{\varkappa_0}$ is proved. Analogously one can prove $\ker B = \Pi_{-\varkappa_0}$. If we use again the density of Π in the subspace

$$\widetilde{\mathbf{C}}_v^{\gamma,0} := \left\{ f \in \mathbf{C}_v :\, \lim_{n\to\infty} n^\gamma E_n^v(f) = 0 \right\}$$

of $\mathbf{C}_v^{\gamma,0}$ ($v = u$ or $v = u_0$) and the mapping properties (2.21) and (2.82), then, for all u of the above form, we obtain

$$\begin{aligned} &\sigma_0^{-1}BA\sigma_0 I = I \;\text{ on }\; \widetilde{\mathbf{C}}_{u_0}^{\gamma,0}\,, \;\text{ if }\; \varkappa_0 \le 0\,, \\ &AB = (A\sigma_0 I)(\sigma_0^{-1}B) = I \;\text{ on }\; \widetilde{\mathbf{C}}_u^{\gamma,0}\,, \;\text{ if }\; \varkappa_0 \ge 0 \end{aligned}$$

from the corresponding assertions of Lemma 2.26. Since $f \in \mathbf{H}$ implies $\sigma_0^{-1}f \in \widetilde{\mathbf{C}}_{u_0}^{\gamma,0}$ and $f \in \widetilde{\mathbf{C}}_u^{\gamma,0}$ for some u and some $\gamma > 0$ (see assertion (v) of Corollary 1.18), we get $BAf = BA\sigma_0\sigma_0^{-1}f = f$ if $\varkappa_0 \le 0$ and $ABf = f$ if $\varkappa_0 \ge 0$. Thus, the assertions of the first part of Theorem 2.11, i.e., the properties

$$\ker A = \sigma_0\Pi_{\varkappa_0}\,, \quad \ker B = \Pi_{-\varkappa_0}\,, \quad BA = I \;\text{ if }\; \varkappa_0 \le 0\,, \quad AB = I \;\text{ if }\; \varkappa_0 \ge 0 \tag{2.85}$$

of the operators $A, B \in \mathbb{L}(\mathbf{H})$ are proved. Now, if $\varkappa_0 \ge 0$ and $g \in \mathbf{H}$, then (2.85) shows that $f = Bg \in \mathbf{H}$ is a solution of $Af = g$ and that the set of all solutions $f \in \mathbf{H}$ is given by

$$Bg + \ker A = \left\{ Bg + \sigma_0\, p :\, p \in \Pi_{\varkappa_0} \right\}.$$

Particularly, $\operatorname{im} A = \mathbf{H}$ if $\varkappa_0 \ge 0$. Now, let $\varkappa_0 < 0$. Clearly, $Af = g$ has a solution $f \in \mathbf{H}$ if and only if $g \in \operatorname{im} A$. In this case, (2.85) shows that f is uniquely determined by $f = Bg$. It remains to prove the assertion about $\operatorname{im} A$. First we remark that, for $g = Af \in \operatorname{im} A$ and $p \in \ker B = \Pi_{-\varkappa_0}$,

$$\begin{aligned} \int_{-1}^1 g(x)p(x)b(x)\mu_0(x)\,dx &= \int_{-1}^1 \left[(afp\,b\,\mu_0)(x) + \frac{1}{\pi}\int_{-1}^1 \frac{(p\,b\,\mu_0)(x)\,(bf)(t)}{t-x}\,dt \right] dx \\ &= \int_{-1}^1 \left[(afp\,b\,\mu_0)(t) + \frac{1}{\pi}\int_{-1}^1 \frac{(p\,b\,\mu_0)(x)\,(bf)(t)}{t-x}\,dx \right] dt \qquad (2.86) \\ &= \int_{-1}^1 (\sigma_0^{-1}Bp)(t)\,(bf)(t)\,dt = 0\,. \end{aligned}$$

Here we used a well known result on the interchange of Lebesgue integral and Cauchy principal value integral ([M1, §28]; see also [PS, Proposition 9.11]). Consequently,

$$\operatorname{im} A \subseteq \mathbf{H} \cap \left(b\,\mu_0 \Pi_{-\varkappa_0}\right)^{\perp},$$

where $M^{\perp} := \left\{g : \int_{-1}^{1} g(x)\,\phi(x)\,dx = 0 \text{ for all } \phi \in M\right\}$. For the proof of the reverse inclusion, we need the assumption $b = \widetilde{b}\,q$ from Theorem 2.11. This enables us to use well known results on the operators

$$\widetilde{A} = qAq^{-1}I = A + (qS - SqI)\widetilde{b}I \quad \text{and} \quad \widetilde{B} = qBq^{-1}I = B + \sigma_0(SqI - qS)\widetilde{b}\mu_0 I\,.$$

For $f \in \mathbf{H}$ we have $\widetilde{b}f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}\cup\widetilde{\mathcal{S}})$ because of $\mathcal{S}\cap\widetilde{\mathcal{S}} = \emptyset$. Hence, the operator $(qS - SqI)\widetilde{b}I$ maps $\mathbf{H}$ into $\Pi_{\deg q}$, since the kernel $\pi^{-1}[q(x) - q(t)]/(t - x)$ of $qS - SqI$ is a polynomial of degree $(\deg q - 1)$ in both variables. This shows that also $\widetilde{A}$ and $\widetilde{B}$ have the properties (2.21) and (2.82), respectively, and that $\widetilde{A}(\sigma_0\Pi) \subseteq \Pi$, $\widetilde{B}(\Pi) \subseteq \sigma_0\Pi$. Moreover, for the restricted operators $\widetilde{A} \in \mathbb{L}(\sigma_0\Pi, \Pi)$ and $\widetilde{B} \in \mathbb{L}(\Pi, \sigma_0\Pi)$ assertions similar to (2.81) hold true, more precisely, A, B, $\Pi_{\varkappa_0}$, and $\Pi_{-\varkappa_0}$ can be replaced by $\widetilde{A}$, $\widetilde{B}$, $q\Pi_{\varkappa_0}$, and $q\Pi_{-\varkappa_0}$ in (2.81) (see [PS, Theorem 9.17 and its proof]). In particular, if we suppose that $\varkappa_0 < 0$ and if we modify the proof of $BA = I$, then we obtain

$$\widetilde{A}, \widetilde{B} \in \mathbb{L}(\mathbf{H}) \quad \text{with} \quad \widetilde{B}\widetilde{A} = I\,.$$

From the known properties of $\widetilde{A}$, acting from the weighted $\mathbf{L}^2$–space $\mathbf{L}^2\big(\widetilde{b}\,\sigma_0\big)$ with scalar product $(f, g)_{\widetilde{b}\,\sigma_0} = \int_{-1}^{1} f\,\overline{g}\,\widetilde{b}\,\sigma_0\,dx$ into the weighted $\mathbf{L}^2$–space $\mathbf{L}^2\big(\widetilde{b}\,\mu_0\big)$, it follows

$$\Pi \cap \left(\widetilde{b}\,\mu_0\Pi_{-\varkappa_0}\right)^{\perp} \subseteq \operatorname{im}\widetilde{A} \tag{2.87}$$

(see [PS, Theorems 9.17]). If we take into account that, for any weight u of the form (2.19), the orthogonal projection from $\mathbf{L}^2\big(\widetilde{b}\,\mu_0\big)$ onto $\Pi_{-\varkappa_0}$ (defined with the help of an $\mathbf{L}^2\big(\widetilde{b}\,\mu_0\big)$–orthonormal basis $\{q_0, \ldots, q_{-\varkappa_0-1}\}$ of $\Pi_{-\varkappa_0}$) can also be considered as an operator from $\mathbf{C}_u$ onto $\Pi_{-\varkappa_0}$ (the integrals $(f, q_j)_{\widetilde{b}\,\mu_0}$ are well defined for $f \in \mathbf{C}_u$), then it is easy to show that every $f \in \widetilde{\mathbf{C}}_u^{\gamma,\delta} \cap \left(\widetilde{b}\,\mu_0\Pi_{-\varkappa_0}\right)^{\perp}$ can be approximated in the norm of $\mathbf{C}_u^{\gamma,\delta}$ by elements of $\Pi \cap \left(\widetilde{b}\,\mu_0\Pi_{-\varkappa_0}\right)^{\perp}$. Together with (2.87) and Corollary 1.18 we obtain

$$\mathbf{H} \cap \left(\widetilde{b}\,\mu_0\Pi_{-\varkappa_0}\right)^{\perp} \subseteq \operatorname{im}\widetilde{A}\,.$$

Now, if $g \in \mathbf{H} \cap \left(b\,\mu_0\Pi_{-\varkappa_0}\right)^{\perp}$, then $gq \in \mathbf{H} \cap \left(\widetilde{b}\,\mu_0\Pi_{-\varkappa_0}\right)^{\perp}$ and, consequently,

$$gq = \widetilde{A}f = qAq^{-1}f \quad \text{for some} \quad f \in \mathbf{H}\,.$$

This implies $q^{-1}f = q^{-1}\widetilde{B}\widetilde{A}f = Bq^{-1}\widetilde{A}f = Bg \in \mathbf{H}$. Hence, $g = Aq^{-1}f \in \operatorname{im} A$. ∎

Proof of Proposition 2.9. Let $p_0 = \min\{1/|\alpha_0|, 1/|\beta_0|\}$ $(1/0 := \infty)$. Then,

$$A, B \in \mathcal{L}(\mathbf{L}^p) \quad \text{for} \quad 1 < p < p_0 \qquad \big(\mathbf{L}^p := \mathbf{L}^p(-1,1)\big)$$

([GK, Theorem I.4.1]). Using the density of Π and $\sigma_0\Pi$ in $\mathbf{L}^p$ (see [MP, Lemma II.3.1]) one can show that the assertions (2.81) of Lemma 2.26 remain true, if A and B are considered as operators in $\mathbf{L}^p$ $(1 < p < p_0)$. Thus, every solution $f \in \mathbf{L}^p$ of $Af = g$ must be of the form $f = Bg + \sigma_0\,p$, $p \in \Pi_{\varkappa_0}$. If $g \in \mathbf{H}$, then this implies $f \in \mathbf{H}$, since $B \in \mathbb{L}(\mathbf{H})$. ∎

2.7.3 Proof of Theorem 2.13

First we prove the mapping property (2.32). In view of Theorem 2.1, we have

$$\sigma^{-1}B_{k,l}\in\mathcal{L}\big(\mathbf{C}_u^0,\mathbf{B}_v\big)\,,\quad\text{where}\quad v=v^{\alpha_0+k,\beta_0+l}u \tag{2.88}$$

for all weights u of the form (2.19) with

$$\left\{\begin{array}{ll}\tau_0>-\beta_0\,, & \text{if}\ \ l=0\\ \tau_0=0\,, & \text{if}\ \ l=1\end{array}\right\},\quad \left\{\begin{array}{ll}\tau_{N+1}>-\alpha_0\,, & \text{if}\ \ k=0\\ \tau_{N+1}=0\,, & \text{if}\ \ k=1\end{array}\right\}. \tag{2.89}$$

Moreover, the decomposition (2.30) of $B_{k,l}$ and the property $\sigma_0^{-1}B(\Pi_n)\subseteq\Pi_{n+\varkappa_0}$ of B (see Lemma 2.26) imply

$$\sigma_0^{-1}B_{k,l}(\Pi_n)\subseteq\Pi_{\max\{n+\varkappa_0,k+l\}}\quad\text{for all}\ \ n\in\mathbb{N}\,.$$

In view of (2.88), the elements of $\sigma^{-1}B_{k,l}(\Pi_n)=v^{-k,-l}\sigma_0^{-1}B_{k,l}(\Pi_n)$ must be integrable, i.e., $\sigma_0^{-1}B_{k,l}(\Pi_n)$ consists of polynomials which vanish in the zeros of $v^{k,l}$. Hence,

$$\sigma^{-1}B_{k,l}(\Pi_n)\subseteq\Pi_{n+\varkappa}\quad\text{for all}\ \ n\in\mathbb{N}\,. \tag{2.90}$$

Together with Proposition 2.5 we obtain

$$\sigma^{-1}B_{k,l}\in\mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta},\mathbf{C}_v^{\gamma,\delta-1}\big)\,,\quad v=v^{\alpha_0+k,\beta_0+l}u$$

for all $\gamma>0$, $\delta\in\mathbb{R}$, and all weights u of the form (2.19) which satisfy (2.89). Now, if $g\in\mathbf{H}_{\mathrm{loc}}^{k,l}(\mathcal{S})$, then there exists a weight u of this type such that gu is Hölder continuous and vanishes in the zeros of u. This implies $g\in\mathbf{C}_u^{\gamma,0}$ for some $\gamma>0$ (see assertion (v) of Corollary 1.18) and, consequently, $\sigma^{-1}B_{k,l}\,g\in\mathbf{C}_v^{\gamma,-1}$. By assertion (i) of Proposition 1.19, $v\sigma^{-1}B_{k,l}\,g=h^{-1}uB_{k,l}\,g$ is Hölder continuous and vanishes in the zeros of v. Thus,

$$B_{k,l}\,g\in\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$$

and (2.32) is proved. To prove (2.33) for $B_{k,l}$ we use that, by Proposition 2.5 and Lemma 2.26,

$$\sigma_0^{-1}B\in\mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta},\mathbf{C}_{u_0}^{\gamma,\delta-1}\big)\,,\quad u_0=v^{\alpha_0,\beta_0}u \tag{2.91}$$

for all $\gamma>0$, $\delta\in\mathbb{R}$, and all weights u of the form (2.19) which satisfy

$$\left\{\begin{array}{ll}\tau_0>-\beta_0\,, & \text{if}\ \ l=0\\ \tau_0=-\beta_0\,, & \text{if}\ \ l=1\end{array}\right\},\quad \left\{\begin{array}{ll}\tau_{N+1}>-\alpha_0\,, & \text{if}\ \ k=0\\ \tau_{N+1}=-\alpha_0\,, & \text{if}\ \ k=1\end{array}\right\}. \tag{2.92}$$

Moreover, since $\mathbf{C}_u^{\gamma,\delta}$ is continuously imbedded into $\mathbf{C}_{\widetilde{u}}^{\widetilde{\gamma},\delta}$, where $\widetilde{\gamma}<\gamma$ and $\widetilde{u}=v^{-\varepsilon,-\varepsilon}u$ (see assertion (vii) of Theorem 1.11), it is easy to show that the second addend of the decomposition (2.30) of $B_{k,l}$ maps $\mathbf{C}_u^{\gamma,\delta}$ into $\sigma_0\Pi_{k+l}$. Together with (2.91) we obtain

$$\sigma_0^{-1}B_{k,l}\in\mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta},\mathbf{C}_{u_0}^{\gamma,\delta-1}\big)\,,\quad u_0=v^{\alpha_0,\beta_0}u\quad\text{for all}\ \ \gamma>0,\ \ \delta\in\mathbb{R}\,. \tag{2.93}$$

Since $\sigma_0^{-1}B_{k,l}$ maps Π into $v^{k,l}\Pi$ (see (2.90)) and since every $g \in \mathbf{C}_u^{\gamma,\delta}$ can be approximated by polynomials in the norm of $\mathbf{C}_u^{\gamma/2,\delta}$ (see assertion (iii) of Theorem 1.11), it is easy to prove the following property of the images of the operator (2.93):

$$\text{If } g \in \mathbf{C}_u^{\gamma,\delta}, \text{ then } u_0\, \sigma_0^{-1} B_{k,l}\, g = h^{-1} u B_{k,l}\, g \text{ vanishes in the zeros of } v^{k,l}. \tag{2.94}$$

Now, let $g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$. In view of Corollary 1.18, (v), there exists an u satisfying (2.92) and some $\gamma > 0$ such that $g \in \mathbf{C}_u^{\gamma,0}$. Thus, $\sigma_0^{-1}B_{k,l}\, g \in \mathbf{C}_{u_0}^{\gamma,-1}$ and, consequently, $u_0\, \sigma_0^{-1} B_{k,l}\, g = h^{-1}uB_{k,l}\, g$ is Hölder continuous (by assertion (i) of Proposition 1.19). Together with (2.94) we obtain

$$B_{k,l}\, g \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$$

and (2.33) is proved for $B_{k,l}$. To prove it for A, we use the property

$$A\sigma_0 I \in \mathcal{L}\big(\mathbf{C}_{u_0}^{\gamma,\delta}, \mathbf{C}_u^{\gamma,\delta-1}\big) \quad (\gamma > 0 \text{ and } \delta \in \mathbb{R} \text{ arbitrary})$$

which holds again for weights u of the form (2.19) satisfying (2.92) (see Proposition 2.5 and Lemma 2.26). If $f \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathbf{H}_{\mathrm{loc}}^{k,l}(\mathcal{S})$ ($f \in v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$ is not needed now), then one can find some u which fulfills (2.92) such that $u_0\, \sigma_0^{-1} f$ is Hölder continuous and vanishes in the zeros of u_0. (Remark that u_0 has no zero in -1 and 1 if $l = 1$ and $k = 1$, respectively.) This implies $\sigma_0^{-1} f \in \mathbf{C}_{u_0}^{\gamma,0}$ for some $\gamma > 0$ (see Corollary 1.18, (v)) and, consequently,

$$Af = A\sigma_0\sigma_0^{-1} f \in \mathbf{C}_u^{\gamma,-1}\,.$$

By assertion (i) of Proposition 1.19 we obtain $Af \in \mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$ and (2.33) is proved. By repeating the considerations after (2.30) we get

$$AB_{k,l}\, g = g \quad \text{for all} \quad g \in A(\mathbf{H}_{\mathrm{loc}}(\mathcal{S})) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S}) \text{ which satisfy (2.29)}\,. \tag{2.95}$$

If $\varkappa_0 \geq 0$, then, by Theorem 2.11, $A(\mathbf{H}_{\mathrm{loc}}(\mathcal{S}))$ can be replaced by $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ in (2.95) and it remains to show that exactly the elements

$$f = B_{k,l}\, g + \sigma\, p\,, \quad p \in \Pi_{\varkappa}$$

are the solutions of $Af = g$ in $\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$ if g belongs to this space and satisfies (2.29), and that the condition (2.29) is necessary for the existence of a solution of $Af = g$ in $\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$. To prove the first assertion, we only have to mention that, by (2.95) and Theorem 2.11, all solutions of $Af = g$ in $\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ are given by

$$f = B_{k,l}\, g + \sigma\, p\,, \quad p \in v^{-k,-l}\Pi_{\varkappa_0}\,,$$

and that $p \in v^{-k,-l}\Pi_{\varkappa_0}$ satisfies $\sigma\, p \in v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})$ if and only if $p \in \Pi_{-\varkappa}$. For the proof of the second assertion we remark that, by (2.90),

$$\Pi_{-\varkappa} \subseteq \ker B_{k,l}\,.$$

Hence, for $p \in \Pi_{-\varkappa}$ and $g = Af \in A\big(\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}_{\mathrm{loc}}^{k,l}(\mathcal{S})\big)$, a slight modification of (2.86) shows that the integral given in (2.29) vanishes.

If $\varkappa_0 < 0$, then the last sentence remains true. Thus, also in this case the existence of a solution $f \in \mathbf{H}_{\rm loc}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\rm loc}(\mathcal{S})$ of $Af = g$ implies that g satisfies (2.29). Together with (2.95) and the uniqueness of the solution (see Theorem 2.11) we see that, if such a solution exists, then it is given by $f = B_{k,l}\,g$. If the additional assumptions on b given in Theorem 2.11 hold true and if $g \in \mathbf{H}_{\rm loc}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\rm loc}(\mathcal{S})$ satisfies (2.29), then formula (2.26) shows that $g \in A(\mathbf{H}_{\rm loc}(\mathcal{S}))$. By (2.95) we obtain that $f = B_{k,l}\,g$ solves $Af = g$ in this case.

Let us summarize: We have proved (2.32), (2.33), and all assertions of Remark 2.14. Clearly, the assertions of Theorem 2.13 follow from Remark 2.14 and mapping property (2.32), since $\mathbf{H}^{k,l}_{\rm loc}(\mathcal{S})$ is a subspace of $\mathbf{H}_{\rm loc}(\mathcal{S}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\rm loc}(\mathcal{S})$. ■

2.7.4 Proof of Theorem 2.21

For weights $v = v^{\mu,\nu}$ with $\mu, \nu \ge -1$ we define

$$\mathbf{C}_{\mu,\nu} := \left\{ f \in \mathbf{C}(-1,1) : \, fv \in \mathbf{C} \text{ (continuous extension)} \right\}.$$

For $f \in \mathbf{C}_{\mu,\nu}$ we introduce the notation

$$\|f\|_{\mu,\nu} := \|fv\|\,, \quad E_n^{\mu,\nu}(f) := \inf\left\{ \|f - p_n\|_{\mu,\nu} : \, p_n \in \Pi_n \cap \mathbf{C}_{\mu,\nu} \right\}$$

and write $f \in \mathbf{C}^{\gamma,\delta}_{\mu,\nu}$ if and only if $\sup_{n\in\mathbb{N}} n^\gamma E_n^{\mu,\nu}(f) \ln^\delta(n+1) < \infty$. We remark that in the case $\mu < 0$ ($\nu < 0$) a function $f \in \mathbf{C}_{\mu,\nu}$ must satisfy $\lim_{x\to1} f(x) = 0$ ($\lim_{x\to-1} f(x) = 0$). Particularly, $P \in \Pi$ belongs to $\mathbf{C}_{\mu,\nu}$ if and only if $P(1) = 0$ if $\mu < 0$, $P(-1) = 0$ if $\nu < 0$.

Lemma 2.27 *Let $\rho, \tau \in \mathbb{R}$ and $\mu \ge \max\{0, -\rho\}$, $\nu \ge \max\{0, -\tau\}$. Then,*

$$v^{\rho,\tau} \in \mathbf{C}^{2\min\{\rho+\mu,\,\tau+\nu\},0}_{\mu,\nu}\,,$$

where $\rho + \mu$ ($\tau + \nu$) can be replaced by an arbitrary positive number if $\rho \in \mathbb{N}_0$ ($\tau \in \mathbb{N}_0$).

Proof. Proposition 1.12 shows that, without loss of generality, we may consider $E_n^{0,\nu}(v^{0,\tau})$ instead of $E_n^{\mu,\nu}(v^{\rho,\tau})$, since $v^{\rho,\tau} = v^{\rho,0}v^{0,\tau}$ and $v^{\rho,0}$ can be treated in a similar way. If $\tau \in \mathbb{N}_0$, then $v^{0,\tau} \in \Pi \subseteq \mathbf{C}^{\gamma,0}_{0,\nu}$ for all $\gamma > 0$. If $\tau = -\nu$, then $v^{0,\tau} \in \mathbf{C}_{0,\nu} = \mathbf{C}^{2(\tau+\nu),0}_{0,\nu}$. Thus, we may assume that

$$\tau \notin \mathbb{N}_0\,, \quad \nu > -\tau\,, \quad \text{and} \quad \nu \ge 0\,.$$

Define $k \in \mathbb{N}_0$ by $k < 2(\nu + \tau) \le k + 1$. Then we have

$$\left\|(v^{0,\tau})^{(k+1)}\varphi^{k+1}v^{0,\nu}\right\|_{\mathbf{C}[-1+h^2,1-h^2]} \le c\left\|v^{0,\nu+\tau-(k+1)/2}\right\|_{\mathbf{C}[-1+h^2,1]} \le c\,h^{2(\nu+\tau)-k-1}\,,$$

$$\left\|(v^{0,\tau})^{(k)}\varphi^{k}v^{0,\nu}\right\|_{\mathbf{C}([-1,-1+h^2]\cup[1-h^2,1])} \le c\left\|v^{0,\nu+\tau-(k/2)}\right\|_{\mathbf{C}([-1,-1+h^2]\cup[1-h^2,1])} \le c\,h^{2(\nu+\tau)-k}$$

for all $h \in (0,1)$. By assertion (ii) of Corollary 1.18 we obtain $v^{0,\tau} \in \mathbf{C}^{2(\tau+\nu),0}_{0,\nu}$ and the lemma is proved. ■

Lemma 2.28 *Let u be a weight of the form (2.19) (where the restriction $\tau_j < 1$ can be replaced by $\tau_j \le 1$) and let $\mu \ge -\tau_{N+1}$, $\nu \ge -\tau_0$. If $f \in \mathbf{C}_u^{\gamma,\delta}$ and $g \in \mathbf{C}_{\mu,\nu}^{\gamma,\delta}$, then*

$$fg \in \mathbf{C}_{u(\mu,\nu)}^{\gamma,\delta}\,, \quad \text{where} \quad u(\mu,\nu) := v^{\mu,\nu}u \quad \text{and} \quad \|fg\|_{u(\mu,\nu),\gamma,\delta} \le c\,\|f\|_{u,\gamma,\delta}\,\|g\|_{v,\gamma,\delta}$$

with some constant $c \ne c(f,g)$.

Proof. Use (1.13) (with $v = v^{\mu,\nu}$). ■

The following lemma is a special case of a more general result which we will prove later (see Theorem 3.5).

Lemma 2.29 *Let $\mu,\nu \in (-1,1)$ and let u be a weight of the form (2.19) with*

$$\mu^- \le \tau_{N+1} < 1-\mu^+\,, \quad \nu^- \le \tau_0 < 1-\nu^+ \qquad \big(\mu^- = \max\{0,-\mu\}\,,\ \mu^+ = \max\{0,\mu\}\,\big)\,.$$

If $h \in v\,\mathbf{C}_v^{\gamma,\delta}$, where $v = v^{\mu,\nu}P$ with

$$P = v^{p,q} \in \big\{v^{0,0}, v^{1,0}, v^{0,1}, v^{1,1}\big\} \text{ such that } 0 \le \mu+p,\, \nu+q < 1\,,$$

then the operator K defined by

$$(Kf)(x) = \int_{-1}^{1} \frac{v^{-\mu,-\nu}(x)h(x) - v^{-\nu,-\nu}(t)h(t)}{t-x}\, f(t)\,dt$$

($x \in (-1,1) \cap \operatorname{supp} u$) belongs to $\mathcal{L}\big(\mathbf{C}_u, \mathbf{C}_{u(\mu,\nu)}^{\gamma,\delta-1}\big)$.

Lemma 2.30 *Let $\rho,\tau \ge 0$, $\gamma \in (0,2)\setminus\{1\}$, and $\delta \le 0$. Further, let*

$$f \in \mathbf{C}^{\gamma,\delta} \text{ if } 0<\gamma<1 \quad \text{and} \quad f' \in \mathbf{C}^{\gamma-1,\delta} \text{ if } 1<\gamma<2\,.$$

If f vanishes in the zeros of $v^{\rho,\tau}$, then $fv^{-\rho,-\tau} \in \mathbf{C}_{\rho,\tau}^{\gamma,\delta}$.

Proof. We may assume that $\rho > 0$ and $\tau > 0$, since it is clear how the following considerations have to be modified if one of the exponents of $v^{\rho,\tau}$ vanishes.

Let $\gamma < 1$ and set $g(t) = f(\cos t)$. Clearly, $f \in \mathbf{C}^{\gamma,\delta}$ implies

$$E_n^T(g) \le c\,\frac{\ln^{-\delta}(n+1)}{n^\gamma}\,, \quad E_n^T(g) := \inf\big\{\|g-g_n\|_{\mathbf{C}[0,2\pi]} : g_n \in \mathcal{T}_n = \operatorname{span}\{\mathrm{e}^{imt}\}_{m=-n+1}^{n-1}\big\}\,.$$

For the norm in the approximation space $\mathbf{C}_{2\pi}^{\gamma,\delta}$ based on $\mathbf{C}_{2\pi} = \{f \in \mathbf{C}(\mathbb{R}) : f = f(\,.\,+2\pi)\}$ (endowed with $\|\,.\,\|_\infty := \|\,.\,\|_{\mathbf{C}[0,2\pi]}$) and $\{\mathcal{T}_n\}$ (which is defined analogously to $\mathbf{C}^{\gamma,\delta}$; see Definition 1.9) we have the equivalence

$$\|g\|_{\gamma,\delta} \sim \|g\|_\infty + \sup_{h\in(0,1]} \frac{K(g,h)}{h^\gamma}\ln^\delta(1+h^{-1})\,, \tag{2.96}$$

where $K(g,h)$ denotes the K-functional (see assertion (vii) of Theorem 1.4) with respect to $\mathbf{X} = \mathbf{C}_{2\pi}$ and

$$\mathbf{W} = \left\{ f \in \mathbf{C}_{2\pi} : \|f\|_{\mathbf{W}} = \|f\|_\infty + |f|_{\mathbf{W}} := \|f\|_\infty + \sup_{t\ne\tau}\left|\frac{f(t)-f(\tau)}{t-\tau}\right| < \infty \right\}.$$

Indeed, for $\mathbf{X} = \mathbf{C}_{2\pi}$, $\mathbf{W}$, and $\mathbf{X}_n = \mathcal{T}_n$ the assumptions (J) and (B) from Theorem 1.4,(vii) are satisfied with $r = 1$ ([DL, Theorem 4.1.1 and Corollary 7.2.4]). Since $\gamma < 1$, also the assumption on $a_n \sim n^\gamma \ln^\delta(n+1)$ from Theorem 1.4,(vii) is satisfied (take $s = \gamma + \varepsilon < 1$). Thus, (2.96) follows from Theorem 1.4 and the analogue of (1.21). In (2.96) we can replace $K(g,h)$ by

$$\omega(g,h) := \sup_{|t-\tau|\le h} |g(t) - g(\tau)| \,,$$

since $\omega(g,h) \le c\, K(g,h) \le c\,(\omega(g,h) + h\,\|g\|_\infty)$ (see [DL, Theorem 6.2.1] and remark that the K-functional $K(g,h;\mathbf{C}_{2\pi},\mathbf{W})$ which is considered in [DL] satisfies $K(g,h;\mathbf{C}_{2\pi},\mathbf{W}) \le K(g,h) \le 2K(g,h;\mathbf{C}_{2\pi},\mathbf{W}) + h\,\|g\|_\infty$, $h \le 1$). If we take into account that $f(1) = 0$, then we obtain

$$|f(x)| = |g(\arccos x) - g(0)| \le \omega(g, \arccos x) \le c\,(\arccos x)^\gamma \ln^{-\delta}\left(1 + \frac{1}{\arccos x}\right).$$

The same holds true with $\arccos x$ replaced by $\pi - \arccos x$ on the right hand side, since $g(\pi) = 0$. We have $\arccos x = \sqrt{2(1-x)} + O(1-x)$ for $x \to 1$ and $\pi - \arccos x = \sqrt{2(1+x)} + O(1+x)$ for $x \to -1$. Consequently,

$$|f(x)| \le c\, v^{\gamma/2,\gamma/2}(x) \ln^{-\delta}\left(1 + \frac{1}{\sqrt{1-x^2}}\right) =: v(x)\,. \tag{2.97}$$

After this preparation it is easy to prove the assertion in the case $\gamma < 1$: Let $p_n, q_n \in \Pi_n$ such that

$$\|v^{-\rho,-\tau} - q_n\|_{\rho+(\gamma/2),\tau+(\gamma/2)} = E_n^{\rho+(\gamma/2),\tau+(\gamma/2)}\big(v^{-\rho,-\tau}\big) \quad \text{and} \quad \|f - p_n\| = E_n(f)\,.$$

The weight $w = v^{\rho,\tau} v$ (v from (2.97)) is a so-called Ditzian-Totik type weight of the class J_∞^* (see [DT] for the definition of J_∞^*). It is well known that, for such weights, $\|P\|_w \sim \|Pw\|_{\mathbf{C}[-1+(2n)^{-2},1-(2n)^{-2}]}$ for all $P \in \Pi_n$ and all $n \in \mathbb{N}$ ([DT, Theorem 8.4.8]). This implies

$$\|P\|_w \le c\,\|P\|_{\rho+(\gamma/2),\tau+(\gamma/2)} \ln^{-\delta}(n+1)\,, \quad P \in \Pi_n\,.$$

Consequently, the embedding operator $A : \Pi \to \mathbf{Y} = \mathbf{C}_w$ satisfies the assumption (1.25) of Corollary 1.23 with $\mathbf{X} = \mathbf{C}_{\rho+(\gamma/2),\tau+(\gamma/2)}$, $\mathbf{X}_n = \Pi_n$, and $a_n = \log_2^{-\delta}(n+1)$. Together with $v^{-\rho,-\tau} \in \mathbf{C}^{\gamma,0}_{\rho+(\gamma/2),\tau+(\gamma/2)}$ (see Lemma 2.27) we obtain

$$\left\|v^{-\rho,-\tau} - q_n\right\|_w \le c\,\frac{\ln^{-\delta}(n+1)}{n^\gamma}\,.$$

Using $|f| \le v$, $v^{-\rho,-\tau} \in \mathbf{C}^{\gamma,0}_{\rho+(\gamma/2),\tau+(\gamma/2)}$, and $\|q_n\|_{\rho,\tau} \sim \left\|v^{\rho,\tau} q_n\right\|_{\mathbf{C}[-1+(2n)^{-2},1-(2n)^{-2}]}$ (since also $v^{\rho,\tau}$ is a Ditzian-Totik weight of the class J_∞^*; see [DT]) we get

$$\begin{aligned}
\left\|f v^{-\rho,-\tau} - p_n\, q_n\right\|_{\rho,\tau} &\le \left\|\left(v^{-\rho,-\tau} - q_n\right) f\,\right\|_{\rho,\tau} + \|q_n\|_{\rho,\tau}\, \|f - p_n\| \\
&\le \left\|v^{-\rho,-\tau} - q_n\right\|_w + c\,\frac{\left\|(q_n - v^{-\rho,-\tau})\, v^{\rho,\tau}\right\|_{\mathbf{C}[-1+(2n)^{-2},1-(2n)^{-2}]} + \|v^{-\rho,-\tau}\|_{\rho,\tau}}{n^\gamma \ln^\delta(n+1)} \\
&\le \left\|v^{-\rho,-\tau} - q_n\right\|_w + c\,\frac{n^\gamma \left\|v^{-\rho,-\tau} - q_n\right\|_{\rho+(\gamma/2),\tau+(\gamma/2)} + 1}{n^\gamma \ln^\delta(n+1)} \le \frac{c}{n^\gamma \ln^\delta(n+1)}\,,
\end{aligned}$$

which shows $E_{2n-1}^{\rho,\tau}(fv^{-\rho,-\tau}) \le c\, n^{-\gamma} \ln^{-\delta}(n+1)$. This implies $fv^{-\rho,-\tau} \in \mathbf{C}_{\rho,\tau}^{\gamma,\delta}$.

Now, let $\gamma \in (1,2)$. In view of Proposition 1.12 and Lemma 2.27, we have $v^{1/2,1/2}f' \in \mathbf{C}^{\gamma-1,\delta}$ and from the assertion in the case $\gamma < 1$ it follows that the first addend on the right hand side of

$$\left(\frac{f}{v^{\rho,\tau}}\right)' = \frac{v^{1/2,1/2}f'}{v^{\rho+(1/2),\tau+(1/2)}} + f\left(\frac{\rho}{v^{\rho+1,\tau}} - \frac{\tau}{v^{\rho,\tau+1}}\right) \tag{2.98}$$

belongs to $\mathbf{C}_{\rho+(1/2),\tau+(1/2)}^{\gamma-1,\delta}$. For the second addend we have

$$\left(\frac{f}{v^{\rho+1,\tau}}\right)' = \frac{f'}{v^{\rho+1,\tau}} + f\left(\frac{\rho+1}{v^{\rho+2,\tau}} - \frac{\tau}{v^{\rho+1,\tau+1}}\right) \in \mathbf{C}_{\rho+1,\tau+1}$$

(since $v^{-1,0}f \in \mathbf{C}$ because of $f \in \mathbf{C}^1[-1,1]$ and $f(1) = 0$) and, analogously, $\left(v^{-\rho,-\tau-1}f\right)' \in \mathbf{C}_{\rho+1,\tau+1}$. Consequently, the second addend of (2.98) belongs to

$$\mathbf{W}\left(v^{\rho+(1/2),\tau+(1/2)},1\right) \subseteq \mathbf{C}_{\rho+(1/2),\tau+(1/2)}^{1,0} \subseteq \mathbf{C}_{\rho+(1/2),\tau+(1/2)}^{\gamma-1,\delta}$$

(see Proposition 1.14). Thus, the left hand side of (2.98) is an element of $\mathbf{C}_{\rho+(1/2),\tau+(1/2)}^{\gamma-1,\delta}$. By Corollary 1.18, (i), this implies $v^{-\rho,-\tau}f \in \mathbf{C}_{\rho,\tau}^{\gamma,\delta}$. ∎

Proof of Theorem 2.21. All assertions except the mapping properties (2.64) and $H \in \mathcal{L}\left(\mathbf{C}_u, v_0\mathbf{C}_{u_0}^{\gamma_0,0}\right)$ ($u_0 = v^{\alpha_0+k,\beta_0+l}u$, $v_0 = v^{\alpha_0+k,\beta_0+l}$) are proved in Section 2.6. It remains to prove (2.64), since all assumptions which are necessary for this mapping property are satisfied if $\gamma > 0$ is chosen small enough and since (2.64) implies $H \in \mathcal{L}\left(\mathbf{C}_u, v_0\mathbf{C}_{u_0}^{\gamma_0,0}\right)$ for some $\gamma_0 > 0$ because of the continuous embedding

$$\frac{1}{\sigma_{-\alpha,-\beta}\, r^2}\mathbf{C}_{u(\alpha,\beta)}^{\gamma,\delta} \subseteq v_0\mathbf{C}_{u_0}^{\gamma_0,0}\,, \quad u(\alpha,\beta) := v^{\alpha,\beta}u$$

(see assertion (v) of Corollary 1.18 and assertion (i) of Proposition 1.19). In view of Corollary 1.18, (i), we have $a, b \in \mathbf{C}^{\gamma,\delta+1}$ also in the case $\gamma > 1$. Thus, it is clear that the part

$$f \longrightarrow -\frac{\widetilde{k}\,\widetilde{l}\,b}{\int_{-1}^{1}\sigma_{\alpha,\beta}(\tau)\,d\tau}\int_{-1}^{1} b(t)\,f(t)\,dt$$

of the operator $\sigma_{-\alpha,-\beta}\,r^2\,H$ belongs to $\mathcal{L}\left(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta+1}\right) \subseteq \mathcal{L}\left(\mathbf{C}_u, \mathbf{C}_{u(\alpha,\beta)}^{\gamma,\delta}\right)$. It remains to consider the operator K defined by

$$(Kf)(x) = \int_{-1}^{1}\frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\,b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\,b(x)}{t-x}\,f(t)\,dt\,.$$

We split it into $K = bK_1 + \widetilde{a}\,\sigma_{-\alpha,-\beta}\,K_2$, where

$$\begin{aligned}(K_1f)(x) &= \int_{-1}^{1}\frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)}{t-x}\,f(t)\,dt\,,\\ (K_2f)(x) &= \int_{-1}^{1}\frac{b(t)-b(x)}{t-x}\,f(t)\,dt\,.\end{aligned}$$

If $b(-1) \neq 0$ and $b(1) \neq 0$, then we apply Lemma 2.29 to K_1 and K_2, where

$$v_1 = v^{\alpha+\widetilde{k},\beta+\widetilde{l}}\,,\;\; h_1 = \widetilde{a}\,\sigma_{-\alpha,-\beta}\,v^{\alpha,\beta}\,, \quad \text{and} \quad v_2 = 1\,,\;\; h_2 = b\,,$$

respectively. From Lemma 2.30 (applied to $f = \widetilde{a}$ and $v^{\rho,\tau} = v^{\alpha+\widetilde{k},\beta+\widetilde{l}}$; remark that $\widetilde{a}(-1) = \widetilde{a}(1) = 0$ because of (2.63)) and Proposition 1.12 it follows that

$$v_1^{-1}(t)\,h_1(t) = \frac{\widetilde{a}(t)}{v^{\alpha+\widetilde{k},\beta+\widetilde{l}}(t)} \left[\mathrm{e}^{-2}(1-t)^{1-t}(1+t)^{1+t}\right]^{\frac{\alpha+\beta-(\operatorname{sign}\alpha+\operatorname{sign}\beta)/2}{2}} \cos(\ell_{\alpha,\beta}(t))$$

belongs to $\mathbf{C}^{\gamma,\delta+1}_{\alpha+\widetilde{k},\beta+\widetilde{l}}$, where we took into account that the last two factors of $v^{-1}h_1$ belong to $\mathbf{W}(1,2) \subseteq \mathbf{C}^{2,0} \subseteq \mathbf{C}^{\gamma,\delta+1}$ (see Proposition 1.14). Consequently,

$$K_1 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(\alpha,\beta)}\big) \quad \text{and} \quad K_2 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u}\big)\,.$$

From $v_1^{-1}h_1 \in \mathbf{C}^{\gamma,\delta+1}_{\alpha+\widetilde{k},\beta+\widetilde{l}}$ it follows $\widetilde{a}\,\sigma_{-\alpha,-\beta} \in \mathbf{C}^{\gamma,\delta+1}_{\alpha,\beta} \subseteq \mathbf{C}^{\gamma,\delta}_{\alpha,\beta}$, since

$$\|(v_1^{-1}h_1 - p_n)v_1\| = \left\|\big(\widetilde{a}\,\sigma_{-\alpha,-\beta} - p_n\,v^{\widetilde{k},\widetilde{l}}\big)v^{\alpha,\beta}\right\| \quad \text{for all} \quad p_n \in \Pi_{n-\widetilde{k}-\widetilde{l}}\,.$$

Together with Lemma 2.28 we obtain

$$bK_1 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(\alpha,\beta)}\big) \quad \text{and} \quad \widetilde{a}\,\sigma_{-\alpha,-\beta}\,K_2 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(\alpha,\beta)}\big)\,.$$

Now, let for example $b(1) = 0$ and $b(-1) \neq 0$. (In the remaining cases the proof is similar.) Instead of the above functions v_i and h_i we take

$$v_1 = v^{\alpha+(\gamma/2),\beta}P_1,\;\; h_1 = \widetilde{a}\,\sigma_{-\alpha,-\beta}\,v^{\alpha+(\gamma/2),\beta} \;\; \text{and} \;\; v_2 = v^{-(\gamma/2),0}P_2,\;\; h_2 = b\,v^{-(\gamma/2),0}\,,$$

$$\text{where} \quad P_1 = \left\{ \begin{array}{ll} v^{1,\widetilde{l}} & \text{if} \quad \alpha+(\gamma/2) < 0 \\ v^{0,\widetilde{l}} & \text{if} \quad \alpha+(\gamma/2) \geq 0 \end{array} \right\} \quad \text{and} \quad P_2 = v^{1,0}\,.$$

From $v^{\gamma/2,0} \in \mathbf{C}^{\gamma,0} \subseteq \mathbf{C}^{\gamma,\delta+1}$ (see Lemma 2.27) and the considerations in the case $b(-1)b(1) \neq 0$ it follows

$$h_1 \in \mathbf{C}^{\gamma,\delta+1} \quad \text{and} \quad h_1(-1) = h_1(1) = 0\,.$$

In view of Lemma 2.30, this implies $v_1^{-1}h_1 \in \mathbf{C}^{\gamma,\delta+1}_{v_1}$ and Lemma 2.29 yields

$$K_1 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(\alpha+(\gamma/2),\beta)}\big)\,.$$

From $b \in \mathbf{C}^{2\gamma,\delta+1}$ and $b(1) = 0$ it follows the existence of polynomials $b_n \in \Pi_n$, $n \in \mathbb{N}$, such that

$$b_n(1) = 0 \quad \text{and} \quad \|b - b_n\| = \left\|v^{-1,0}b - v^{-1,0}b_n\right\|_{1,0} \leq \frac{c}{n^{2\gamma}\ln^{\delta+1}(n+1)}$$

(take $b_n = p_n - p_n(1)$, where p_n is the polynomial of best approximation to b). Thus, $v^{-1,0}b \in \mathbf{C}^{2\gamma,\delta+1}_{1,0}$, which implies $v^{-1,0}b \in \mathbf{C}^{\gamma,\delta+1}_{v_2} \subseteq \mathbf{C}^{\gamma,\delta}_{v_2}$ (see assertion (vii) of Theorem 1.11). If we take into account that

$$\left\|(v^{-1,0}b - p_n)v_2\right\| = \left\|b - v^{1,0}p_n\right\|_{-\gamma/2,0} \quad \text{for all} \quad p_n \in \Pi_{n-1}\,,$$

then we see that $b \in \mathbf{C}^{\gamma,\delta}_{-\gamma/2,0}$ and together with Lemma 2.28 we obtain

$$b\,K_1 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(\alpha,\beta)}\big)\,.$$

We have just seen that $v_2^{-1}h_2 = v^{-1,0}b \in \mathbf{C}^{\gamma,\delta+1}_{v_2}$. Thus, by Lemma 2.29,

$$K_2 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(-\gamma/2,0)}\big)\,.$$

From $v_1^{-1}h_1 \in \mathbf{C}^{\gamma,\delta+1}_{v_1}$ it follows $\widetilde{a}\,\sigma_{-\alpha,-\beta} \in \mathbf{C}^{\gamma,\delta+1}_{\alpha+(\gamma/2),\beta} \subseteq \mathbf{C}^{\gamma,\delta}_{\alpha+(\gamma/2),\beta}$, since

$$\|(v_1^{-1}h_1 - p_n)v_1\| = \big\|\big(\widetilde{a}\,\sigma_{-\alpha,-\beta} - p_n\,P_1\big)v^{\alpha+(\gamma/2),\beta}\big\| \quad \text{for all} \quad p_n \in \Pi_{n-\deg P_1}\,.$$

Together with Lemma 2.28 we obtain

$$\widetilde{a}\,\sigma_{-\alpha,-\beta}\,K_2 \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}^{\gamma,\delta}_{u(\alpha,\beta)}\big)$$

and the theorem is proved. ■

2.8 Notes and comments

2.1. Theorem 2.1 was first proved for the case of Jacobi weights u and v in [LR] and, for more special cases of weights, in [CMR]. For power weights u and v Theorem 2.1 can be found in [L3, Section 5]. There it is even proved that, under the additional assumption that a and b_1 are piecewise continuous, the images Af, $f \in \mathbf{C}^0_u$, multiplied by $v[\mathcal{M}]$ are piecewise continuous. In order to simplify the proof we have not considered the weighted piecewise continuity of the images Af in the present paper. This is justified by the fact that such a result is not needed in our applications of Theorem 2.1. For the case of the Cauchy singular integral operator on $(-\infty,\infty)$ results similar to Theorem 2.1 can be found in [DDM] and [DD], where certain weighted moduli of smoothness are used to define appropriate analogues of the norm $\|f\|^*_{u,0}$. Of course, other types of weights are used in the case of the unbounded interval considered in [DDM, DD].

2.2. For the special case of the operator $A_\sigma = A\sigma I$ with A from Section 2.3 and σ from Theorem 2.13 (compare (2.49)) and Jacobi weights $u, v \in \mathbf{C}$ with $v^{-1} \in \mathbf{L}^1(-1,1)$ and $u/v = v^{\alpha_0+k,\beta_0+l}$ the assertion $A_\sigma \in \mathcal{L}\big(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-1}_v\big)$ of Proposition 2.5 (with v instead of $v[\mathcal{M}]$, since Corollary 2.3 can be applied to $S\sigma I$ and $\widetilde{f} = bf$, $f \in \mathbf{C}^0_u$, if $b(\pm 1) = 0$) can be found in [JL1]. For the case of power weights the same mapping property of A_σ can be concluded from [L3, Corollary 5.5]. In certain special cases of operators with constant coefficients and Jacobi weights u, v one can even prove $A_\sigma \in \mathcal{L}\big(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta}_v\big)$; see [MRT, Proof of Theorem 3.1]. Generalizations of Proposition 2.8 can be found in [L3].

2.3 and 2.4. The representation of the locally Hölder continuous solutions of (2.16) given in Theorem 2.11 is a special case of the corressponding result of Muschelischwili [M1, § 98]. Since it is hard to translate the general considerations given in [M1] for equations on curves in $\mathbb{C}$ to the special case of the interval, we have preferred to conclude the assertions of Theorems 2.11 and 2.13 from the $\mathbf{L}^2$–theory given in [PS, Section 9] just for the case of equations on $[-1,1]$.

2.5 and 2.6. The operators $A_{\alpha,\beta}$ are well known in the cases when $\alpha+\beta \in \mathbb{Z}$. These special operators play an important role in the numerical analysis for Cauchy singular integral equations with constant coefficients a and b; see, e.g., [BHS1, BHS2, C, JL2]. They are also used as a tool for the investigation of approximation methods of the type (2.50) for equations with variable coefficients; see the references given before (2.50). As far as we know, the operators $A_{\alpha,\beta}$ with arbitrary $(\alpha,\beta) \in (-1,1) \setminus \{0\}$ are used in the present paper for the first time in order to investigate Cauchy singular integral equations with variable coefficients and approximation methods for them. Hence, Theorem 2.21 is a new result.

Chapter 3

Weakly singular integral operators on [-1,1]

In all of what follows we consider an integral operator K on $(-1,1)$,

$$(Kf)(x) = \int_{-1}^{1} k(x,t)\, f(t)\, dt\,, \quad x \in (-1,1)\,,$$

where the kernel function $k(x,t)$ is defined and continuous on $[-1,1]^2 \setminus N$, N a set of measure zero. More precisely, we suppose that there is a weight

$$v \in \mathbf{C}\,,\ v \geq 0 \quad \text{with} \quad \text{meas}\,(\text{supp}_* v) = 2 \quad (\,\text{supp}_* v = \{x \in [-1,1] : v(x) \neq 0\}\,) \tag{3.1}$$

and a second weight, defined on a subset $D \subseteq [-1,1]$ of Lebesgue measure 2,

$$w \in \mathbf{C}(D) \quad \text{with} \quad w(x) > 0 \ \text{ for all } \ x \in D\,, \tag{3.2}$$

such that

$$g(x,t) = (x-t)\, v(x)\, k(x,t)\, v(t)\, w(t) \in \mathbf{C}\big([-1,1]^2\big) \quad \text{and} \quad g(t,t) = 0. \tag{3.3}$$

This means that $k(x,t)$ is defined and continuous on $[\,\text{supp}_* v\,]^2 \setminus \{(x,t) : x = t \text{ or } t \notin D\}$ and that $g(x,t)$ can be continuously extended onto $[-1,1]^2$, where the extension vanishes on the diagonal $\{x = t\}$ of $[-1,1]^2$.

We will ask for additional conditions which ensure that, for a possibly big number of weights u (defined and positive on a subset of $[-1,1]$ having the Lebesgue measure 2), the operator K maps the spaces

$$\mathbf{L}^\infty_u := \big\{f : fu \in \mathbf{L}^\infty(-1,1)\big\} \quad \big(\text{endowed with the norm } \|f\|_u := \|fu\|_{\mathbf{L}^\infty(-1,1)}\big)$$

into spaces of the type $\mathbf{C}^{\gamma,\delta}_{\varrho}$, where ϱ may depend on u, v, and w. We distinguish the cases of kernels $k(x,t)$ without moving singularities (singularities in $x = t$ are called moving) and kernels with moving singularities.

3.1 Kernels without moving singularities

Let us consider the case of a kernel $k \in \mathbf{C}(\operatorname{supp}_* v \times D)$. More precisely, instead of (3.3) we suppose that even

$$h(x,t) := v(x)\,k(x,t)\,w(t) \in \mathbf{C}\big([-1,1]^2\big)\,. \tag{3.4}$$

Thus, if v has zeros in $x = x_i$ and w has zeros in $t = t_j$, then $k(x,t)$ may have singularities on the lines $\{x_i\} \times [-1,1]$ and $[-1,1] \times \{t_j\}$.

The following theorem shows that K is a smoothing integral operator if $k(x,t)$ is smooth in x, i.e., $k(\,.\,,t) \in \mathbf{C}_v^{\gamma,\delta}$.

Theorem 3.1 *Let (3.1), (3.2), (3.4) be satisfied and suppose that $k(\,.\,,t) \in \mathbf{C}_v^{\gamma,\delta}$ for all $t \in D$ ($\gamma > 0$ and $\delta \in \mathbb{R}$ fixed), where*

$$\sup_{t\in D} \big\|k(\,.\,,t)\,w(t)\big\|_{v,\gamma,\delta} < \infty\,. \tag{3.5}$$

Then, $K \in \mathcal{L}\big(\mathbf{L}_u^\infty, \mathbf{C}_v^{\gamma,\delta}\big)$ for all measurable $u : D \to (0,\infty)$ with $(u\,w)^{-1} \in \mathbf{L}^1(-1,1)$.

Remark 3.2 *In the proof of Theorem 3.1 we will even show that, under the above assumptions, $K \in \mathcal{L}\big(\mathbf{L}^1_{w^{-1}}, \mathbf{C}_v^{\gamma,\delta}\big)$, where*

$$\mathbf{L}^1_{w^{-1}} := \big\{f \,:\, fw^{-1} \in \mathbf{L}^1(-1,1)\big\} \qquad \big(\textit{with norm } \|f\| = \|fw^{-1}\|_{\mathbf{L}^1(-1,1)}\big)\,.$$

Moreover, similar theorems hold true for arbitrary spaces of the type $(\mathbf{C}_v)^{\mathcal{A}}_\infty(\{\Pi_n\})$ (see Definition 1.1) instead of $\mathbf{C}_v^{\gamma,\delta}$. We have restricted on $\mathbf{L}_u^\infty$ and $\mathbf{C}_v^{\gamma,\delta}$ since these are the spaces of interest in our later investigations of approximation methods for Cauchy singular integral equations.

Now we will show that (3.4) can be replaced by a slightly weaker assumption. Namely, it is sufficient that $v(x)\,k(x,t)\,w(t)$ is continuous in x and piecewise continuous in t. Clearly, if w is a power weight and $\varrho(t) = \prod_i |x - \xi_i|$, where ξ_i are the points in which $v(x)\,k(x,t)\,w(t)$ may have jumps as a function in t, then every power weight u with $(u\,w)^{-1} \in \mathbf{L}^1$ satisfies also $(u\,w\,\varrho^\varepsilon)^{-1} \in \mathbf{L}^1$ and we can take $w\,\varrho^\varepsilon$ instead of w to generalize Theorem 3.1 in the case of such special weights u and w. But one can also consider general weights. To prove this, we first mention that, if (3.4) and (3.5) are satisfied, then we have even

$$\sup_{t\in[-1,1]} \big\|v^{-1}(\,.\,)h(\,.\,,t)\big\|_{v,\gamma,\delta} < \infty\,. \tag{3.6}$$

This is shown in the proof of Theorem 3.1 (see (3.14)). Now, if $h(x,t)$ is only piecewise continuous in t,

$$h(x,t) \in \mathbf{C}\big([-1,1] \times [\xi_j, \xi_{j+1}]\big)\,,\ j = 0,\ldots,R \quad (-1 = \xi_0 < \xi_1 < \ldots < \xi_{R+1} = 1) \tag{3.7}$$

(here we suppose that D is chosen small enough such that $\{\xi_j\}_{j=0}^{R+1} \subseteq [-1,1] \setminus D$, and by $h(x,t) \in \mathbf{C}\big([-1,1] \times [\xi_j, \xi_{j+1}]\big)$ we mean that h, defined on $\operatorname{supp}_* v \times [D \cap (\xi_j, \xi_{j+1})]$, possesses a continuous extension on $[-1,1] \times [\xi_j, \xi_{j+1}]$), then we define

$$k_j(x,t) = v^{-1}(x)\,h_j(x,t)\,w^{-1}(t)\,, \quad \text{where} \quad h_j(x,t) = \begin{cases} h(x,t)\,, & t \in (\xi_j, \xi_{j+1})\,, \\ h(x, \xi_{j+1} - 0)\,, & t \ge \xi_{j+1}\,, \\ h(x, \xi_j + 0)\,, & t \le \xi_j\,. \end{cases}$$

If (3.7) and (3.5) are satisfied, then $\sup_{t\in D\cap(\xi_j,\xi_{j+1})} \left\|v^{-1}(\,.\,)h_j(\,.\,,t)\right\|_{v,\gamma,\delta} < \infty$ and (3.6) (which holds similar for functions on $[-1,1]\times[\xi_j,\xi_{j+1}]$ instead of functions on $[-1,1]^2$) shows that even $\sup_{t\in[\xi_j,\xi_{j+1}]} \left\|v^{-1}(\,.\,)h_j(\,.\,,t)\right\|_{v,\gamma,\delta} < \infty$, i.e.,

$$\sup_{t\in[-1,1]} \left\|v^{-1}(\,.\,)h_j(\,.\,,t)\right\|_{v,\gamma,\delta} < \infty\,.$$

Now we take into account that the operator of multiplication by the characteristic function χ_j of $[\xi_j,\xi_{j+1}]$ is bounded in $\mathbf{L}_u^\infty$ ($\mathbf{L}_{w^{-1}}^1$) and that K can be decomposed as follows,

$$K = \sum_{j=0}^{R} K_j\,\chi_j I\,, \quad \text{where} \quad (K_j f)(x) = \int_{-1}^{1} k_j(x,t)\,f(t)\,dt\,. \tag{3.8}$$

Since Theorem 3.1 (Remark 3.2) can be applied to each operator K_j, we obtain the following.

Corollary 3.3 *Let v and w be weights of the type (3.1) and (3.2), respectively. If $h(x,t) = v(x)\,k(x,t)\,w(t)$ satisfies (3.7) instead of (3.4) and if (3.5) holds, then the assertions of Theorem 3.1 and Remark 3.2 remain true.*

Remark 3.4 *Corollary 3.3 is useful if the smoothness of $k(\,.\,,t)w(t)$ in the near of ±1 is higher than $\mathbf{C}_v^{\gamma,\delta}$. More precisely, let $v\in\mathbf{C}[a,b]$, $[a,b]\supseteq[-1,1]$, with $\operatorname{meas} D(v) = b-a$, where $D(v) = \{x\in[a,b] : v(x)>0\}$, and suppose that $k(x,t)\in\mathbf{C}\big(D(v)\times D(w)\big)$. If $h(x,t) = v(x)\,k(x,t)\,w(t)$ satisfies (3.7), where we take x from $[a,b]$ instead of $[-1,1]$, and if (3.5) holds true with $\|\,.\,\|_{\mathbf{C}_v^{\gamma,\delta}[a,b]}$ instead of $\|\,.\,\|_{v,\gamma,\delta}$, where $\mathbf{C}_v^{\gamma,\delta}[a,b] = \mathbf{C}_v[a,b]_\infty^{\mathcal{A}}(\{\Pi_n\})$ with $\mathcal{A}$ from Remark 1.10 and $\mathbf{C}_v[a,b] = \{f : fv\in\mathbf{C}[a,b]\}$, then*

$$K\in\mathcal{L}\big(\mathbf{L}_{w^{-1}}^1,\mathbf{C}_v^{\gamma,\delta}[a,b]\big) \quad \Big(\textit{particularly, } K\in\mathcal{L}\big(\mathbf{L}_u^\infty,\mathbf{C}_v^{\gamma,\delta}[a,b]\big) \textit{ if } (uw)^{-1}\in\mathbf{L}^1\Big),$$

since Corollary 3.3, transformed onto $[a,b]^2$, can be applied to $k(x,t)\,\chi_{[-1,1]}(t)$ (and $w := 1$ outside $[-1,1]$).

We mention that, for $[a,b]\supset[-1,1]$, $f\in\mathbf{C}_v^{\gamma,\delta}[a,b]$ means indeed more than $f\in\mathbf{C}_v^{\gamma,\delta}$ (i.e., not every $\mathbf{C}_v^{\gamma,\delta}$–function can be extended to a $\mathbf{C}_v^{\gamma,\delta}[a,b]$–function). For example, $f\in\mathbf{C}^{\gamma,0}[-1-\varepsilon,1+\varepsilon]$, $\gamma\notin\mathbb{N}$, implies $f\in\mathbf{H}^\gamma([-1,1])$ (Proposition 1.19 with $[-1-\varepsilon,1+\varepsilon]$ instead of $[-1,1]$), while $f\in\mathbf{C}^{\gamma,0}$ is also possible for $f\notin\mathbf{H}^\gamma([-1,1])$ (see Corollary 1.18).

3.2 Kernels with singularities in $x = t$

Now we consider kernels $k(x,t)$ for which $(t-x)\,v(x)\,k(x,t)\,v(t)\,w(t)$ can be extended to a continuous function on $[-1,1]^2$ which vanishes on the diagonal $\{x=t\}$. More precisely, we suppose that

$$\begin{aligned} &k(x,t) = \frac{1}{(t-x)\,w(t)}\left[\frac{P(x)\,h(x,t)}{v(x)} - \frac{P(t)\,h(t,t)}{v(t)}\right], \quad \text{where} \quad h\in\mathbf{C}\big([-1,1]^2\big) \quad \text{and} \\ &P(x) = \prod_{i=1}^{N}(x-x_i)^{k_i} \quad \text{with} \quad -1\le x_1<x_2<\ldots<x_N\le 1 \ (N\in\mathbb{N}) \quad \text{and} \quad k_i\in\mathbb{N}_0\,. \end{aligned} \tag{3.9}$$

Here we assume that $v \in \mathbf{C}$ is a power weight with $vP^{-1}, v^{-1}P \in \mathbf{L}^1(-1,1)$, i.e.,

$$v(x) = \prod_{i=1}^{N} |x - x_i|^{\beta_i} \quad \text{with} \quad \beta_i \geq 0 \;\text{ and }\; k_i - 1 < \beta_i < k_i + 1 \;\text{ for all }\; i \tag{3.10}$$

(zeros x_i of v which are no zeros of P are simply included in the above representation of P by setting $k_i = 0$) and that w satisfies (3.2).

Theorem 3.5 *Let v and w be weights of the form (3.10) and (3.2), respectively, and let $k(x,t)$ satisfy (3.9). If $v^{-1}(\,.\,)\, h(\,.\,,t) \in \mathbf{C}_v^{\gamma,\delta}$ for all $t \in D$ ($\gamma > 0$ and $\delta \in \mathbb{R}$ fixed), where*

$$\sup_{t \in D} \left\| v^{-1}(\,.\,)\, h(\,.\,,t) \right\|_{v,\gamma,\delta} < \infty\,, \tag{3.11}$$

then $K \in \mathcal{L}\big(\mathbf{L}_u^\infty, \mathbf{C}_{uvw/|P|}^{\gamma,\delta-1}\big)$ for all

$$u(x) = w^{-1}(x) \prod_{i=1}^{N} |x - x_i|^{\alpha_i} \quad \textit{with} \quad \max\{0,\, k_i - \beta_i\} \leq \alpha_i < 1 + \min\{0,\, k_i - \beta_i\}\,. \tag{3.12}$$

($Kf \in \mathbf{C}_{uvw/|P|}^{\gamma,\delta-1}$ for $f \in \mathbf{L}_u^\infty$ is meant in the sense that $(Kf)(x)$ is defined for $x \in \operatorname{supp} uvw$ and can be extended to a continuous function on $\operatorname{supp}(uvw/|P|)$ belonging to $\mathbf{C}_{uvw/|P|}^{\gamma,\delta-1}$).

As in the proof of Corollary 3.3 one can show that the continuity assumption on $h(x,\,.\,)$ can be weakened: If only (3.7) holds true, then we define $k_j(x,t)$ by replacing h by h_j in (3.9) (h_j is defined after (3.7)). Applying Theorem 3.5 to each operator K_j of the decomposition (3.8) we obtain the following.

Corollary 3.6 *If the condition $h(x,t) \in \mathbf{C}\big([-1,1]^2\big)$ in (3.9) is replaced by (3.7), then Theorem 3.5 remains true.*

As in Remark 3.4, the above Corollary can be used to consider the images $(Kf)(x)$ of certain singular integral operators with moving singularities also for $x \notin [-1,1]$:

Remark 3.7 *Let $k(x,t)$ be also defined for $x \in [-b,-1) \cup (1,b]$ ($b > 1$) and suppose that (3.9) holds true with some $h \in \mathbf{C}([-b,b] \times [-1,1])$ satisfying (3.11) with $\|\,.\,\|_{\mathbf{C}_v^{\gamma,\delta}[-b,b]}$ instead of $\|\,.\,\|_{v,\gamma,\delta}$. If u and v are weights of the form (3.10) and (3.12), respectively, then*

$$K \in \mathcal{L}\big(\mathbf{L}_u^\infty, \mathbf{C}_{uvw/|P|}^{\gamma,\delta-1}[-b,b]\big)\,.$$

(Remark that, by (3.12), uw and, hence, uvw are well-defined on $[-b,b]$.)

3.3 Other types of moving singularities

Let us shortly discuss the case of kernels $k(x,t)$ with singularities on nonlinear curves $C \subseteq [-1,1]^2$. We start with an example.

Example 3.8 *Let* $k(x,t) = \dfrac{2t^2(t-1)}{\sqrt{|x-t^2|\,|x-t|}}$ *and write*

$$\begin{aligned}
(Kf)(x) &= \int_{-1}^{1} \frac{\sqrt{|x-t|}}{\operatorname{sign}(x-t)}\, \frac{2tf(t)}{\sqrt{|x-t^2|}}\, dt - \int_{-1}^{1} \frac{\sqrt{|x-t^2|}}{\operatorname{sign}(x-t^2)}\, \frac{2tf(t)}{\sqrt{|x-t|}}\, dt \\
&= \int_{0}^{1} \frac{\sqrt{|x-\sqrt{t}|}}{\operatorname{sign}(x-\sqrt{t})}\, \frac{f\big(\sqrt{t}\big)}{\sqrt{|x-t|}}\, dt - \int_{0}^{1} \frac{\sqrt{|x+\sqrt{t}|}}{\operatorname{sign}(x+\sqrt{t})}\, \frac{f\big(-\sqrt{t}\big)}{\sqrt{|x-t|}}\, dt \\
&\quad - \int_{-1}^{1} \frac{\sqrt{|x-t^2|}}{\operatorname{sign}(x-t^2)}\, \frac{2tf(t)}{\sqrt{|x-t|}}\, dt\,.
\end{aligned}$$

Theorem 3.5 (together with Corollary 3.6) can be applied to each addend of this decomposition and we obtain

$$K \in \mathcal{L}\big(\mathbf{L}^\infty, \mathbf{C}^{1/2,-1}\big)\,.$$

This example underlines the following fact: Often it is possible, after an appropriate decomposition of the kernel and variable substitutions in the resulting integrals, to apply Theorem 3.5 and Corollary 3.6 to obtain mapping properties of an operator K the kernel of which has singularities on a finite number of curves

$$x = \phi_1(t)\,,\; x = \phi_2(t)\,,\; \ldots\,,\; x = \phi_k(t)$$

(ϕ_i: continuous functions on $[-1,1]$) which have at most finitely many points of intersection in $[-1,1]^2$. More precisely, the following decomposition of the kernel may be used if

$$g(x,t) = v(x)\, k(x,t)\, w(t)\, [x-\phi_1(t)] \ldots [x-\phi_k(t)]$$

is a continuous function which vanishes on the curves $x = \phi_i(t)$:

$$k(x,t) = \sum_{i=1}^{k} \frac{v^{-1}(x)\, g(x,t)\, \Phi_i(t)}{x-\phi_i(t)}\,, \quad \text{where} \quad \Phi_i(t) = \frac{1}{w(t)} \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{1}{\phi_i(t)-\phi_j(t)}\,.$$

In the resulting decomposition of $(Kf)(x)$ into a sum of integrals one can try to use the substitutions $\tau = \phi_i(t)$ to obtain operators to which Theorem 3.5 or Corollary 3.6 can be applied. Of course, this requires additional assumptions on the functions ϕ_i. We will not give a corresponding general theorem, since its precise formulation would be very involved. We think that, if one has a special operator, then it is better to examine the details for this concrete example.

3.4 Proofs

3.4.1 Proof of Theorem 3.1

Lemma 3.9 *If* $h \in \mathbf{C}([-1,1]^2)$ *and*

$$C := \sup_{t\in D} \big\| v^{-1}(\,.\,) h(\,.\,,t) \big\|_{v,\gamma,\delta} < \infty\,,$$

then there are functions $h_n(x,t)$ *of the form*

$$h_n(x,t) = v(x)\big[c_0^{(n)}(t) + c_1^{(n)}(t)\,x + \ldots + c_{n-1}^{(n)}(t)\,x^{n-1}\big] \quad \textit{with} \;\; c_i^{(n)} \in \mathbf{C} \tag{3.13}$$

and a constant $c > 0$ *such that, for all* $n \in \mathbb{N}_0$*,*

$$\|h - h_n\|_{\mathbf{C}([-1,1]^2)} \le \frac{c}{(n+1)^\gamma \ln^\delta(n+2)}\,. \tag{3.14}$$

Proof. Let $n \in \mathbb{N}_0$ and let $P_n(\,.\,,t) \in \Pi_n$ ($t \in D$ fixed) denote a polynomial of best approximation to $v^{-1}(\,.\,)h(\,.\,,t)$ in the norm of $\mathbf{C}_v$. Then,

$$\|h(\,.\,,t) - v(\,.\,)P_n(\,.\,,t)\| = E_n^v\big(v^{-1}(\,.\,)h(\,.\,,t)\big) \le \frac{C}{(n+1)^\gamma \ln^\delta(n+2)}\,.$$

Further, choose $\delta_n > 0$ such that

$$|h(x,t) - h(x,t_0)| \le \frac{1}{(n+1)^\gamma \ln^\delta(n+2)} \quad \text{for} \;\; t, t_0, x \in [-1,1] \;\; \text{with} \;\; |t - t_0| < \delta_n.$$

Then we obtain

$$|h(x,t) - v(x)P_n(x,t_0)| \le \frac{1+C}{(n+1)^\gamma \ln^\delta(n+2)} \tag{3.15}$$

for all $(x,t) \in [-1,1]^2$ and $t_0 \in D$ with $|t - t_0| < \delta_n$. Now we choose numbers $t_k \in D$, $k = 1, \ldots, m$ (t_k and m depending on n), such that

$$\begin{aligned} &-1 < t_1 < t_2 < \ldots < t_m < 1 \quad \text{and} \\ &\max\{t_1 + 1,\, t_2 - t_1\,, \ldots\,,\, t_m - t_{m-1},\, 1 - t_m\} < \delta_n\,. \end{aligned}$$

Then we define

$$h_n(x,t) = v(x) \sum_{k=0}^{m+1} P_n(x, \widetilde{t}_k)\, B_k(t)\,, \quad \widetilde{t}_k = \begin{cases} t_k\,, & 1 \le k \le m\,, \\ t_1\,, & k = 0\,, \\ t_m\,, & k = m+1\,, \end{cases}$$

where $B_k\,,\; k = 0, 1, \ldots, m+1$, are the linear B–splines with respect to the partition $t_0 = -1,\, t_1, \ldots, t_m,\, t_{m+1} = 1$, i.e.,

$$\begin{aligned} B_0(t) &= \frac{\max\{0,\, t_1 - t\}}{t_1 + 1}\,, \quad B_{m+1}(t) = \frac{\max\{0,\, t - t_m\}}{1 - t_m}\,, \\ B_k(t) &= \max\left\{0\,,\, \min\left\{\frac{t - t_{k-1}}{t_k - t_{k-1}}\,,\, \frac{t_{k+1} - t}{t_{k+1} - t_k}\right\}\right\}\,, \quad k = 1, \ldots, m\,. \end{aligned}$$

Clearly, $h_n(x,t)$ is a function of the required form. Moreover, $\sum_{k=0}^{m+1} B_k = 1$ on $[-1,1]$ and, consequently,

$$h(x,t) - h_n(x,t) = \sum_{k=0}^{m+1} \big[h(x,t) - v(x)P_n(x,\widetilde{t}_k)\big]\, B_k(t)\,.$$

If we take into account that this sum has at most two non-zero addends for every t and that the distance between $\widetilde{t}_k$ and any point $t \in [-1,1]$ of the support of $B_k(t)$ is less than δ_n, then, in view of (3.15), we obtain (3.14) (with $c = 2 + 2C$). ∎

Proof of Theorem 3.1. Let $h(x,t)$ be the continuous extension of $v(x)\,k(x,t)\,w(t)$. If we write

$$(Kf)(x) = \frac{1}{v(x)} \int_{-1}^{1} h(x,t)\, f(t)\, \frac{dt}{w(t)}\,, \tag{3.16}$$

then it becomes clear that K is a bounded linear operator from $\mathbf{L}^1_{1/w}$ into $\mathbf{B}_v$, where the operator norm satisfies the estimate

$$\big\|K\big\|_{\mathbf{L}^1_{1/w}\to\mathbf{B}_v} \le \|h\|_{\mathbf{C}([-1,1]^2)}\,. \tag{3.17}$$

Now we take the functions $h_n(x,t)$ from Lemma 3.9 and define K_n by

$$(K_n f)(x) := \frac{1}{v(x)} \int_{-1}^{1} h_n(x,t)\, f(t)\, \frac{dt}{w(t)}\,. \tag{3.18}$$

If we take into account that $h_n(x,t)$ has the form (3.13), then we obtain

$$K_n \in \mathcal{L}\big(\mathbf{L}^1_{1/w}, \Pi_n\big) \quad \text{for all } \ n \in \mathbb{N}_0\,. \tag{3.19}$$

Now we consider $K - K_n$. Again this is an operator of the type (3.16) (replace $h(x,t)$ by $h(x,t) - h_n(x,t)$) and (3.17) together with (3.14) show that

$$\big\|K - K_n\big\|_{\mathbf{L}^1_{1/w}\to\mathbf{B}_v} \le \|h - h_n\|_{\mathbf{C}([-1,1]^2)} \le \frac{c}{(n+1)^\gamma \ln^\delta(n+2)}\,, \quad n \in \mathbb{N}_0\,. \tag{3.20}$$

Particularly, every image function Kf is the $\mathbf{B}_v$–limit of the polynomials $K_n f \in \Pi_n$ (see (3.19)) and, hence, must be an element of $\mathbf{C}_v$. Thus, (3.20) and (3.19) imply

$$K \in \mathcal{L}\big(\mathbf{L}^1_{1/w}, \mathbf{C}^{\gamma,\delta}_v\big)\,.$$

If $(u\,w)^{-1} \in \mathbf{L}^1(-1,1)$, then $\mathbf{L}^\infty_u$ is continuously embedded into $\mathbf{L}^1_{1/w}$ and we obtain $K \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{C}^{\gamma,\delta}_v\big)$. ∎

3.4.2 Proof of Theorem 3.5

Lemma 3.10 *If $0 \le \beta_i < 1$ for all i, then there is some constant $c \ne c(g,x)$ such that*

$$\int_{-1}^{1} \left|\frac{g(x,t)}{t-x}\right| \frac{dt}{v(t)} \le \frac{c}{v(x)} \left(\|g(x,\,.\,)\|_\infty + \int_{-1}^{1} \left|\frac{g(x,t)}{t-x}\right| dt \right)$$

for all $x \in \operatorname{supp} v$ and all $g : [-1,1]^2 \to \mathbb{C}$ with $g(x,\,.\,) \in \mathbf{L}^\infty(-1,1)$, $x \in \operatorname{supp} v$.

Proof. Let $v \not\equiv 1$ (for $v \equiv 1$ the assertion is trivial). We have

$$v^{-1}(x) \sim |x - x_1|^{-\beta_1} + \ldots + |x - x_N|^{-\beta_N}\,.$$

(Write the right hand side as a fraction or consider the cases $x \in I_j$, where I_j are neighborhoods of the points x_j.) Thus, we do not lose generality if we suppose that

$$v(x) = |x-y|^\beta\,, \quad \text{where} \quad y \in [-1,1] \;\text{ and }\; \beta \in (0,1) \;\text{ are fixed}\,.$$

Now, a slight modification of the proof of (2.68) yields the assertion. ■

Proof of Theorem 3.5. In view of assertion (i) of Proposition 1.19, there exists some $\eta \in (0,1]$ such that $h(\,.\,,t) \in \mathbf{H}^\eta([-1,1])$ for all $t \in D$, where $\sup_{t\in D} \|h(\,.\,,t)\|_{\mathbf{H}^\eta} < \infty$. Consequently,

$$|h(x,t) - h(t,t)| \le c\,|x-t|^\eta \quad \text{for all} \;\; (x,t) \in [-1,1]\times D\,. \tag{3.21}$$

Together with the Hölder continuity of v, the estimate $|f(t)| \le c\,\|f\|_u\, u^{-1}(t)$ a.e., and Lemma 3.10 (applied to $uvw/|P|$ instead of v) this shows that the absolute value of

$$\begin{aligned}(Kf)(x) &= \\ &\int_{-1}^1 \frac{v^{-1}(x)P(x)h(x,t) - v^{-1}(t)P(t)h(t,t)}{t-x}\, f(t)\,\frac{dt}{w(t)} = \\ &\frac{1}{v(x)}\int_{-1}^1 \frac{P(x)-P(t)}{t-x}\,h(x,t)\,f(t)\,\frac{dt}{w(t)} + \frac{1}{v(x)}\int_{-1}^1 \frac{v(t)h(x,t)-v(x)h(t,t)}{t-x}\,f(t)\,\frac{P(t)\,dt}{(vw)(t)}\end{aligned} \tag{3.22}$$

can be estimated by $c\,\|f\|_u\,(uv^2w)^{-1}(x)\,|P(x)|$. Thus, $K \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{B}_{uv^2w/|P|}\big)$. Now we approximate $h(x,t)$ by $h_n(x,t)$ from Lemma 3.9. If we replace $h(x,t)$ by $h_n(x,t)$ in (3.22), then we obtain an operator K_n which maps $\mathbf{L}^\infty_u$ into $\Pi_{n+\deg P-1}$, since

$$\frac{v^{-1}(x)P(x)h_n(x,t) - v^{-1}(t)P(t)h_n(t,t)}{t-x} = \sum_{i=0}^{n-1} c_i^{(n)}(t)\,\frac{P(x)\,x^i - P(t)\,t^i}{t-x}$$

is a polynomial of degree less than $n+\deg P-1$ in x the coefficients of which are continuous functions in t. Now we will estimate the norm of $K-K_n$, first in $\mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{B}_{uv^2w}\big)$ and then even in $\mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{B}_{uvw/|P|}\big)$. For this aim, we introduce the intervals

$$I_{n,x} = \left[x - \frac{1+x}{n^s}\,,\, x + \frac{1-x}{n^s}\right],$$

where $s > 0$ is some sufficiently large constant. (The following considerations will show how big s must be.) Let $\chi_{n,x}(t)$ be the characteristic function of $I_{n,x}$ and let $f \in \mathbf{L}^\infty_u$.

Then, for all $x \in \operatorname{supp}_* uvw$,

$$\begin{aligned}
&\big|[(K-K_n)f](x)\big| \\
&\quad \le c\,\|f\|_u \left[\int_{-1}^{1} \left|\frac{\chi_{n,x}(t)\,[v^{-1}(x)\,P(x)\,h(x,t)-v^{-1}(t)\,P(t)\,h(t,t)]}{t-x}\right| \frac{dt}{(uw)(t)}\right. \\
&\qquad\qquad + \int_{-1}^{1} \left|\frac{\chi_{n,x}(t)\,[v^{-1}(x)\,P(x)\,h_n(x,t)-v^{-1}(t)\,P(t)\,h_n(t,t)]}{t-x}\right| \frac{dt}{(uw)(t)} \\
&\qquad\qquad + \frac{|P(x)|}{v(x)} \int_{-1}^{1} \left|\frac{[1-\chi_{n,x}(t)]\,[h(x,t)-h_n(x,t)]}{t-x}\right| \frac{dt}{(uw)(t)} \\
&\qquad\qquad \left. + \int_{-1}^{1} \left|\frac{1-\chi_{n,x}(t)}{t-x}\right| \frac{\big|P(t)\,[h(t,t)-h_n(t,t)]\big|}{(uvw)(t)}\,dt \quad\right] \\
&\quad =: c\,\|f\|_u \left[I_1 + I_2 + \frac{|P(x)|}{v(x)}\,I_3 + I_4\right].
\end{aligned}$$

We may assume that the exponent η in (3.21) is also a Hölder exponent of v. Then, by the analogue of decomposition (3.22),

$$I_1 \le \frac{c}{v(x)} \int_{-1}^{1} \chi_{n,x}(t)\,\frac{dt}{(uw)(t)} + \frac{c}{v(x)} \int_{-1}^{1} \chi_{n,x}(t)\,|t-x|^{\eta-1}\,\frac{|P(t)|\,dt}{(uvw)(t)}\,.$$

Lemma 3.10 can be applied to both addends, where we take $g(x,t) = \chi_{n,x}(t)\,|t-x|$ for the first one and $g(x,t) = \chi_{n,x}(t)\,|t-x|^{\eta}$ for the second. If we take into account that $|t-x| \le 2/n^s$ for $t \in I_{n,x}$, then we obtain

$$\begin{aligned}
I_1 &\le \frac{c}{(uvw)(x)} \left[\frac{1}{n^s} + \int_{I_{n,x}} dt\right] + \frac{c\,|P(x)|}{(uv^2w)(x)} \left[\frac{1}{n^{s\eta}} + \int_{I_{n,x}} |t-x|^{\eta-1}\,dt\right] \\
&\le c\,\frac{n^{-s\eta}}{(uv^2w)(x)} \le c\,\frac{n^{-\gamma}\ln^{-\delta}(n+1)}{(uv^2w)(x)}\,, \qquad (3.23)
\end{aligned}$$

supposed that $s > \gamma/\eta$. To estimate I_2 we use that

$$\begin{aligned}
\chi_{n,x}(t) \left|P(x)\,\frac{h_n(x,t)}{v(x)} - P(t)\,\frac{h_n(t,t)}{v(t)}\right| &\le \big\|\,[P(\,.\,)\,v^{-1}(\,.\,)\,h_n(\,.\,,t)]'\,\big\|\,\chi_{n,x}(t)\,|x-t| \\
&\le \frac{2\,\big\|\,[P(\,.\,)\,v^{-1}(\,.\,)\,h_n(\,.\,,t)]'\,\big\|}{n^s}\,.
\end{aligned}$$

If we take into account that $P_t := P(\,.\,)\,v^{-1}(\,.\,)\,h_n(\,.\,,t)$ is a polynomial of degree less than $n + \deg P$ and that, by (1.53) and (1.12), $\|P_t'\| \le c\,n^2\|P_t\|$ and $\|P_t\| \le c\,n^{\mu}\|P_t\|_v$ ($\mu = \mu(v) > 0$ some constant), then we obtain

$$\begin{aligned}
\big\|\,[P(\,.\,)\,v^{-1}(\,.\,)\,h_n(\,.\,,t)]'\,\big\| &\le c\,n^{2+\mu}\|h_n(\,.\,,t)\| \\
&\le c\,n^{2+\mu}\big(\|h_n-h\|_{\mathbf{C}([-1,1]^2)} + \|h\|_{\mathbf{C}([-1,1]^2)}\big) \le c\,n^{2+\mu}\,.
\end{aligned}$$

Together with Lemma 3.10 we get

$$I_2 \leq \frac{c}{(uw)(x)} \left(\frac{n^{2+\mu}}{n^s} + n^{2+\mu} \int_{I_{n,x}} dt \right) \leq c\, \frac{n^{2+\mu-s}}{(uw)(x)} \leq c\, \frac{n^{-\gamma} \ln^{-\delta}(n+1)}{(uv^2w)(x)}\,, \tag{3.24}$$

supposed that $s > \gamma+\mu+2$. In I_3 and I_4 we estimate $|h(x,t)-h_n(x,t)|$ and $|h(t,t)-h_n(t,t)|$, respectively, by

$$\|h - h_n\|_{\mathbf{C}([-1,1]^2)} \leq \frac{c}{n^\gamma \ln^\delta(n+1)}$$

(see Lemma 3.9). By Lemma 3.10, the remaining integrals are bounded by

$$\frac{c}{(uw)(x)} \left[1 + \int_{[-1,1]\setminus I_{n,x}} \frac{dt}{|t-x|} \right] \quad \text{and} \quad c\, \frac{|P(x)|}{(uvw)(x)} \left[1 + \int_{[-1,1]\setminus I_{n,x}} \frac{dt}{|t-x|} \right],$$

respectively. The last integral behaves like $\ln n$ and we obtain

$$I_3 \leq \frac{c}{(uw)(x)}\, \frac{1}{n^\gamma \ln^{\delta-1}(n+1)}\,, \qquad I_4 \leq c\, \frac{|P(x)|}{(uvw)(x)}\, \frac{1}{n^\gamma \ln^{\delta-1}(n+1)}\,. \tag{3.25}$$

Thus, $\|Kf - K_n f\|_{uv^2w} \leq c\, \|f\|_u\, n^{-\gamma} \ln^{1-\delta}(n+1)$. Particularly,

$$K_n \to K \quad \text{in} \;\; \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{B}_{uv^2w}\big)\,. \tag{3.26}$$

Now we will prove that we have even convergence in $\mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{B}_{uvw/|P|}\big)$. For this aim, we first remark that, for every $f \in \mathbf{L}^\infty_u$, the polynomial $(K_{2n} - K_n)f \in \Pi_{2n+\deg P-1}$ satisfies

$$\begin{aligned} &\big\|(K_{2n} - K_n)f\big\|_{uvw/|P|} \\ &\qquad \leq c\, \big\|(uvw/|P|)(x)\, [(K_{2n} - K_n)f](x)\big\|_{\mathbf{C}\left([-1,1]\setminus \bigcup_i (x_i - Cn^{-2}, x_i + Cn^{-2})\right)}\,, \end{aligned} \tag{3.27}$$

where $C > 0$ is a sufficiently small constant (see (1.69)). For $x \notin \bigcup_i \big(x_i - Cn^{-2}, x_i + Cn^{-2}\big)$ we have

$$v(x)\, |P(x)| = \prod_{i=1}^N |x - x_i|^{\beta_i + k_i} \geq c\, n^{-2\max\{\beta_1+k_1, \ldots, \beta_N+k_N\}}\,.$$

Consequently, if the number s which appears in the estimates (3.23) and (3.24) is chosen large enough, then we obtain

$$I_1, I_2 \leq c\, \frac{|P(x)|}{(uvw)(x)}\, \frac{1}{n^\gamma \ln^{\delta-1}(n+1)}\,, \qquad x \notin \bigcup_i \big(x_i - Cn^{-2}, x_i + Cn^{-2}\big)\,.$$

The estimates (3.25) for I_3 and I_4 remain true. Thus, for $x \notin \bigcup_i \big(x_i - Cn^{-2}, x_i + Cn^{-2}\big)$,

$$\big|[(K_{2n} - K_n)f](x)\big| \leq \big|[(K_{2n} - K)f](x)\big| + \big|[(K - K_n)f](x)\big| \leq \frac{c\,|P(x)|}{(uvw)(x)}\, \frac{1}{n^\gamma \ln^{\delta-1}(n+1)}$$

and together with (3.27) we get

$$\big\|(K_{2n} - K_n)f\big\|_{uvw/|P|} \leq c\, \frac{\|f\|_u}{n^\gamma \ln^{\delta-1}(n+1)} \quad \text{for all} \;\; n \in \mathbb{N}\,. \tag{3.28}$$

In view of (3.26) we have, for all $f \in \mathbf{L}_u^\infty$,

$$Kf - K_n f = \sum_{j=0}^{\infty} (K_{2^{j+1}n} - K_{2^j n}) f \quad \text{(convergence in } \mathbf{B}_{uv^2w}) \,. \tag{3.29}$$

But (3.28) shows that this series is even absolutely convergent in $\mathbf{B}_{uvw/|P|}$:

$$\sum_{j=0}^{\infty} \left\| K_{2^{j+1}n} - K_{2^j n} f \right\|_{uvw/|P|} \le c \sum_{j=0}^{\infty} \frac{2^{-\gamma j} n^{-\gamma} \, \|f\|_u}{[j \ln 2 + \ln(n+1)]^{\delta-1}} \le \frac{c\, \|f\|_u}{n^\gamma \ln^{\delta-1}(n+1)} \,. \tag{3.30}$$

(Here we used $\ln(n+1) \le j \ln 2 + \ln(n+1) \le 2j \ln(n+1)$, $j \ge 1$.) From (3.29) and (3.30) it follows $Kf \in \mathbf{B}_{uvw/|P|}$ and

$$\|Kf - K_n f\|_{uvw/|P|} \le c \, \frac{\|f\|_u}{n^\gamma \ln^{\delta-1}(n+1)} \,.$$

Since $K_n f \in \Pi_{n+\deg P-1}$, we conclude $Kf \in \mathbf{C}_{uvw/|P|}$ and

$$E_{n+\deg P-1}^{uvw/|P|}(Kf) \le c \, \frac{\|f\|_u}{n^\gamma \ln^{\delta-1}(n+1)} \le c \, \frac{\|f\|_u}{(n+\deg P)^\gamma \ln^{\delta-1}(n+\deg P+1)} \,, \quad n \in \mathbb{N} \,.$$

Clearly, $E_m^{uvw/|P|}(Kf) \le c\, \|f\|_u \, (m+1)^{-\gamma} \ln^{1-\delta}(m+2)$ is also true for $m < \deg P$, since $\|Kf\|_{uvw/|P|} \le \|Kf - K_1 f\|_{uvw/|P|} + \|K_1 f\|_{uvw/|P|} \le c\, \|f\|_u + c\, \|K_1 f\|_{uv^2w} \le c\, \|f\|_u$. ■

3.5 Notes and comments

3.1. Theorem 3.1 is well known in the case of Jacobi weights u and v; see [JL1, Proposition 4.12]. For general weights it can be found in [L1, Section 3.3].

3.2. The special case $v = P = 1$, $u = w^{-1} v^{\rho,\tau}$ of Theorem 3.5 can be found in [JL1, Proposition 4.13]. For the case $P = 1$, $v = v^{\rho_1,\tau_1}$, $u = v^{\rho_2,\tau_2} w^{-1}$ a result similar to Theorem 3.5 was proved in [L1, Proposition 3.37]. In the case of power weights v and uw Theorem 3.5 is a new result.

Chapter 4

Approximation spaces in the numerical analysis of operator equations

From now on we will investigate the weighted uniform convergence of approximation methods for integral equations of the type

$$(A+K)f = g\,, \tag{4.1}$$

where A is a Cauchy singular integral operator with Hölder continuous coefficients as in Section 2.3 and K is a weakly singular integral operator as in Chapter 3. As explained in Section 2.6, we will first regularize equation (4.1) with the help of a left $\mathbf{C}_u$–regularizer of A and then we will consider approximation methods for the resulting regularized equation. In the present chapter we give a general approach to the study of stability and convergence of approximation methods for regularized operator equations, where we turn from the concrete integral operators A and K to arbitrary operators on a Banach space $\mathbf{X}$ or subspaces of it. This theory will be the basis of our later investigations of Cauchy singular integral equations.

4.1 Stability of approximation methods for regularized equations

Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces and let

$$\{0\} = \mathbf{X}_0 \subseteq \mathbf{X}_1 \subseteq \mathbf{X}_2 \subseteq \dots\,, \quad \{0\} = \mathbf{Y}_0 \subseteq \mathbf{Y}_1 \subseteq \mathbf{Y}_2 \subseteq \dots$$

be finite-dimensional linear subspaces of $\mathbf{X}$ and $\mathbf{Y}$, respectively. To avoid trivialities, we assume that not all $\mathbf{X}_n$ and not all $\mathbf{Y}_n$ are equal to $\{0\}$. The corresponding approximation spaces (see Definition 1.1) are denoted by

$$\mathbf{X}_q^{\mathcal{A}} = \mathbf{X}_q^{\mathcal{A}}\big(\{\mathbf{X}_n\}\big) \quad \text{and} \quad \mathbf{Y}_q^{\mathcal{A}} = \mathbf{Y}_q^{\mathcal{A}}\big(\{\mathbf{Y}_n\}\big)\,,$$

respectively. Let us recall that $\mathcal{A} = \{a_n\}_{n=0}^{\infty}$ has to satisfy

$$0 = a_0 < 1 = a_1 < a_2 < a_3 < \dots\,, \tag{4.2}$$

$$\lim_{n\to\infty} a_n = \infty\,, \tag{4.3}$$

$$a_{n+1} \le K a_n \;\text{ for all }\; n \in \mathbb{N}\,,\;\text{ where } K > 1 \text{ is some constant}. \tag{4.4}$$

Now we consider a fixed sequence $\mathcal{A}$ with the above properties and an operator

$$A \in \mathcal{L}\big(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y}\big)\,.$$

In other words, A is the unique bounded extension onto $\mathbf{X}_1^{\mathcal{A}}$ of an operator $A \in \mathbb{L}(\bigcup \mathbf{X}_n, \mathbf{Y})$ satisfying

$$\|A\|_{\mathbf{X}_n \to \mathbf{Y}} \le \operatorname{const} a_n\,, \quad n \in \mathbb{N} \tag{4.5}$$

(see assertion (i) of Theorem 1.21 and Remark 1.22). Further, let us given a second operator

$$K \in \mathcal{L}\big(\mathbf{X}, \mathbf{Y}_\infty^{\mathcal{AB}}\big) \qquad \big(\mathcal{AB} := \{a_n b_n\}\big)\,, \tag{4.6}$$

where $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ is a fixed sequence of numbers $b_n \ge 0$ satisfying (4.2)–(4.4) (with a_n replaced by b_n). Particularly, $K \in \mathcal{K}(\mathbf{X}, \mathbf{Y})$ in view of assertion (i) of Theorem 1.4. (Later we will see that it is important that in the mapping property (4.6) there appears a sequence in the image space which converges faster to infinity than $\mathcal{A}$. For this reason we have written this sequence in the form of a product $\mathcal{AB}$.) We consider the operator equation

$$(A + K)f = g\,, \tag{4.7}$$

where $g \in \mathbf{Y}_\infty^{\mathcal{AB}}$ is given and $f \in \mathbf{X}_1^{\mathcal{A}}$ is looked for. In general it is not a good idea to study (4.7) as an operator equation in the pair $(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y})$ since, because of the following remark, we cannot expect that $A + K$ is a Fredholm operator in $\mathcal{L}\big(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y}\big)$.

Remark 4.1 *We have not supposed that (4.5) is optimal, i.e., it may happen that there are (unknown) numbers $\widetilde{a}_n$ satisfying (4.2)–(4.4) such that $\widetilde{a}_n/a_n \to 0$ and (4.5) holds with $\widetilde{a}_n$ instead of a_n. In this case, $A \in \mathcal{L}\big(\mathbf{X}_1^{\widetilde{\mathcal{A}}}, \mathbf{Y}\big)$ and $\mathbf{X}_1^{\mathcal{A}}$ is compactly embedded into $\mathbf{X}_1^{\widetilde{\mathcal{A}}}$ (see assertion (v) of Theorem 1.4), implying $A \in \mathcal{K}\big(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y}\big)$ and, consequently, $A + K \in \mathcal{K}\big(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y}\big)$.*

In order to regularize equation (4.7), i.e., to transform it into an equation with an operator of the form "identity plus compact" on the left hand side, we suppose that we have given an appropriate left regularizer $\widehat{A}$ of A. If one wants to study stability and convergence of approximation methods for regularized operator equations, then it turns out that, under the additional assumption

$$A \in \mathcal{L}\big(\mathbf{X}_1^{\mathcal{A}^2}, \mathbf{Y}_1^{\mathcal{A}}\big) \qquad \big(\mathcal{A}^2 := \{a_n^2\}\big)\,,$$

it is reasonable to define the notion of a left regularizer of A in dependence of two sequences $\mathcal{B}$ and $\mathcal{C}$ as follows. (Later we will take just the sequence $\mathcal{B}$ from (4.6). $\mathcal{C}$ can be viewed as a "smoothness parameter" for $\widehat{A}g$, $g \in \mathbf{Y}_\infty^{\mathcal{AB}}$.)

Definition 4.2 *Let $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y}) \cap \mathcal{L}(\mathbf{X}_1^{\mathcal{A}^2}, \mathbf{Y}_1^{\mathcal{A}})$. An operator $\widehat{A} : \mathbf{Y}_1^{\mathcal{A}} \to \mathbf{X}$ is called left $(\mathcal{B}, \mathcal{C})$–regularizer of A if*

(i) $\mathcal{B}$ and $\mathcal{C}$ satisfy (4.2)–(4.4) (with a_n replaced by b_n and c_n, respectively),

(ii) $\widehat{A} \in \mathcal{L}(\mathbf{Y}_1^{\mathcal{A}}, \mathbf{X}) \cap \mathcal{L}(\mathbf{Y}_\infty^{\mathcal{AB}}, \mathbf{X}_\infty^{\mathcal{C}})$, where $\mathcal{AB} = \{a_n b_n\}$,

(iii) there exists an $H \in \mathcal{L}(\mathbf{X}, \mathbf{X}_\infty^{\mathcal{C}})$ such that $\widehat{A}Af = f + Hf$ for all $f \in \bigcup \mathbf{X}_n$.

We have introduced the sequence $\mathcal{AB}$ in the formulation of the second mapping property in (ii) because of the following reason: In general, $\widehat{A}$ is only defined on $\mathbf{Y}_1^{\mathcal{A}}$. Thus, if we consider $\widehat{A}$ on another approximation space, then this space should be continuously embedded into $\mathbf{Y}_1^{\mathcal{A}}$. This is the case for $\mathbf{Y}_\infty^{\mathcal{AB}}$ if $\{a_n^\varepsilon b_n^{-1}\}$ is almost decreasing for some $\varepsilon > 0$ (see assertion (ii) of Theorem 1.21). For the determination of $\mathcal{C}$, the following remark can be helpful.

Remark 4.3 *If $\mathcal{B} = \mathcal{C}$ and if, for some $\varepsilon > 0$,*

$$\{a_n^\varepsilon b_n^{-1}\} \text{ is almost decreasing} \quad \text{and} \quad A(\mathbf{X}_n) \subseteq \mathbf{Y}_n\,,\ \widehat{A}(\mathbf{Y}_n) \subseteq \mathbf{X}_n\,,\ n \in \mathbb{N}\,,$$

then the properties $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}^2}, \mathbf{Y}_1^{\mathcal{A}})$ and $\widehat{A} \in \mathcal{L}(\mathbf{Y}_\infty^{\mathcal{AB}}, \mathbf{X}_\infty^{\mathcal{B}})$ can be omitted in Definition 4.2, since in this case they follow automatically from $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y})$ and $\widehat{A} \in \mathcal{L}(\mathbf{Y}_1^{\mathcal{A}}, \mathbf{X})$ (see assertion (ii) of Theorem 1.21).

Although, in our later applications, we will choose $\mathbf{Y}_n$ in such a way that, in general, $A(\mathbf{X}_n) \not\subseteq \mathbf{Y}_n$ and $\widehat{A}(\mathbf{Y}_n) \not\subseteq \mathbf{X}_n$ (since we do not study spectral methods; see Section 2.6), Remark 4.3 will be useful for parts of respective representations of the operators A and $\widehat{A}$ (see, for example, (2.58)).

The following Proposition shows that equation (4.7) can be regularized with the help of $\widehat{A}$ if the conditions (i)–(iii) of Definition 4.2 are satisfied.

Proposition 4.4 *If $\widehat{A}$ is a left $(\mathcal{B}, \mathcal{C})$–regularizer of $A \in \mathcal{L}(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y}) \cap \mathcal{L}(\mathbf{X}_1^{\mathcal{A}^2}, \mathbf{Y}_1^{\mathcal{A}})$, then*

$$\widehat{A}Af = f + Hf \quad \text{for all } f \in \mathbf{X}_1^{\mathcal{A}^2}\,. \tag{4.8}$$

Moreover, if $g \in \mathbf{Y}_\infty^{\mathcal{AB}}$ and $K \in \mathcal{L}(\mathbf{X}, \mathbf{Y}_\infty^{\mathcal{AB}})$, then every solution $f \in \mathbf{X}_1^{\mathcal{A}^2}$ of (4.7) is also a solution of the so-called regularized equation

$$\big(I + H + \widehat{A}K\big)f = \widehat{A}g\,. \tag{4.9}$$

Particularly, $f = \widehat{A}(g - Kf) - Hf \in \mathbf{X}_\infty^{\mathcal{C}}$ for all solutions $f \in \mathbf{X}_1^{\mathcal{A}^2}$ of (4.7).

The proof of this proposition is left to the reader. We only remark that the density of $\bigcup \mathbf{X}_n$ in $\mathbf{X}_1^{\mathcal{A}^2}$ (see assertion (ii) of Theorem 1.4) can be used to prove (4.8) and that the second assertion can be obtained by applying $\widehat{A}$ from the left to both sides of (4.7).

Now it is justified to consider (4.9) (as operator equation in $\mathbf{X}$) instead of the initial equation (4.7). We mention that, under the assumptions of Proposition 4.4, the operator $I + H + \widehat{A}K$ on the left hand side of (4.9) is a Fredholm operator of index 0 in $\mathbf{X}$, since

$$H, \widehat{A}K \in \mathcal{L}(\mathbf{X}, \mathbf{X}_\infty^{\mathcal{C}}) \subseteq \mathcal{K}(\mathbf{X})\,. \tag{4.10}$$

Now we investigate approximation methods for (4.9) which are of the following type:

$$f_n \in \mathbf{X}_n : \ \big[I + P_n(H_n + \widehat{A} L_n K_n)\big] f_n = P_n \widehat{A} L_n g \,. \tag{4.11}$$

Here we suppose that

$$K_n \in \mathcal{L}(\mathbf{X}_n, \mathbf{Y}) \quad \text{and} \quad H_n \in \mathcal{L}(\mathbf{X}_n, \mathbf{X}) \quad (n \in \mathbb{N}) \tag{4.12}$$

are certain approximations of the operators K and H, respectively (later we will be more precise) and that, for every $n \in \mathbb{N}$,

$$L_n \in \mathcal{L}(\mathbf{Y}, \mathbf{Y}_n) \ \text{ is a projection from } \mathbf{Y} \text{ onto } \mathbf{Y}_n \,, \tag{4.13}$$

$$P_n \in \mathcal{L}(\mathbf{X}, \mathbf{X}_n) \ \text{ is a projection from } \mathbf{X} \text{ onto } \mathbf{X}_n \,, \tag{4.14}$$

Remark 4.5 *Later we consider the Cauchy singular integral equation (4.1) and the regularizer $\widehat{A}$ from Theorem 2.21. As approximation $(K_n f)(x)$ and $(H_n f)(x)$ we take quadrature rules for the integrals $(Kf)(x)$ and $(Hf)(x)$, respectively, and L_n, P_n are certain Lagrangian interpolation operators. This principle difference between the two factors of the approximations $P_n H_n$ of H and $L_n K_n$ of K, respectively, is the reason why we do not consider simply $H_n \in \mathcal{L}(\mathbf{X}_n)$ and $K_n \in \mathcal{L}(\mathbf{X}_n, \mathbf{Y}_n)$ instead of $P_n H_n$ and $L_n K_n$, respectively, in the general form (4.11) of our approximation methods. Moreover, we have already mentioned that it is known how the operator $\widehat{A}$ from Theorem 2.21 acts on polynomials. For this reason we have not replaced $\widehat{A}$ by an approximation $\widehat{A}_n$ in (4.11).*

It is well known that the first step in the proof of convergence estimates for approximation methods

$$f_n \in \mathbf{X}_n : \ A_n f_n = g_n \qquad (\, A_n \in \mathcal{L}(\mathbf{X}_n) \,)$$

is the verification of the stability of the method. By stability we mean that, for all sufficiently large $n \ge n_0$, the operators A_n are invertible in $\mathbf{X}_n$, where

$$\sup_{n \ge n_0} \big\| A_n^{-1} \big\|_{\mathcal{L}(\mathbf{X}_n)} < \infty \qquad (\, \mathbf{X}_n \text{ endowed with the norm of } \mathbf{X} \,) \,. \tag{4.15}$$

In the following main theorem of this section we give a list of conditions which are sufficient for the stability of the method (4.11).

Theorem 4.6 *Let $\widehat{A}$ be a left $(\mathcal{B}, \mathcal{C})$–regularizer of $A \in \mathcal{L}\big(\mathbf{X}_1^{\mathcal{A}}, \mathbf{Y}\big) \cap \mathcal{L}\big(\mathbf{X}_1^{\mathcal{A}^2}, \mathbf{Y}_1^{\mathcal{A}}\big)$, where*

$$\big\{ a_n^{\varepsilon}\, b_n^{-1} \big\} \ \text{ is almost decreasing for some } \varepsilon > 0 \,.$$

Further, let $K \in \mathcal{L}\big(\mathbf{X}, \mathbf{Y}_\infty^{\mathcal{A}\mathcal{B}}\big)$ and let the operators K_n, H_n, L_n, P_n from (4.12)–(4.14) satisfy the following additional assumptions, where $n \ge n_0 \ne n_0(n, f_n)$ and $c \ne c(n, f_n)$:

(i) $\displaystyle\inf_{h_n \in \ker L_n} \big\| (K - K_n) f_n - h_n \big\|_{\mathbf{Y}} \le c\, a_n^{-1} \left[b_n^{-1} + c_n^{-1} \| L_n \|_{\mathcal{L}(\mathbf{Y})}^{-1} \right] \| f_n \|_{\mathbf{X}}$ *for all* $f_n \in \mathbf{X}_n$,

(ii) $\displaystyle\inf_{h_n \in \ker P_n} \big\| (H - H_n) f_n - h_n \big\|_{\mathbf{X}} \le c\, c_n^{-1} \| f_n \|_{\mathbf{X}}$ *for all* $f_n \in \mathbf{X}_n$,

(iii) $\displaystyle\lim_{n \to \infty} \left[c_n^{-1} + b_n^{-1} \| L_n \|_{\mathcal{L}(\mathbf{Y})} \right] \| P_n \|_{\mathcal{L}(\mathbf{X})} = 0.$

If $(I + H + \widehat{A} K) f = 0$ has only the trivial solution in $\mathbf{X}$, then the method (4.11) is stable.

Remark 4.7 *It is important that we take the infima in assumptions (i) and (ii), since this yields certain discrete semi-norms of* $(K-K_n)f_n$ *and* $(H-H_n)f_n$ *which are much weaker than* $\|(K-K_n)f_n\|_{\mathbf{Y}}$ *and* $\|(H-H_n)f_n\|_{\mathbf{X}}$*, respectively. For example, if* $\mathbf{Y}=\mathbf{C}$ *and if* L_n *is an interpolation operator with respect to knots* $x_1,\dots,x_n\in(-1,1)$*, i.e.,* $\ker L_n=\{h\in\mathbf{C}:h(x_i)=0$ *for all* $i\}$*, then* $\inf\limits_{h\in\ker L_n}\|f-h\|=\max\limits_i|f(x_i)|$*. (The infimum is reached for* $h=f-S(f)$*, where* $S(f)\in\mathbf{C}$ *denotes the linear spline which interpolates* f *in the knots* x_i *and which vanishes in* ±1*.)*

4.2 Convergence of approximation methods for regularized equations

Let the assumptions of Theorem 4.6 be satisfied, suppose that $(I+H+\widehat{A}K)f=0$ has only the trivial solution in $\mathbf{X}$, and let the right hand side g of (4.7) belong to $\mathbf{Y}_\infty^{\mathcal{AB}}$. Then we know that the operator $I+H+\widehat{A}K$ is invertible in $\mathcal{L}(\mathbf{X})$ (since, by (4.10), it is a Fredholm operator of index 0) and that, for $n\ge n_0$, the approximative operators $I+P_n(H_n+\widehat{A}L_nK_n)$ are invertible in $\mathcal{L}(\mathbf{X}_n)$, where the norms of the inverses are uniformly bounded. Thus, there exist uniquely determined solutions

$$f^*\in\mathbf{X}\quad\text{and}\quad f_n^*\in\mathbf{X}_n\ \ (n\ge n_0)$$

of (4.9) and (4.11), respectively. Of course, we want to know whether f_n^* converges to f^*. More precisely, we are interested in possibly good estimates for the error

$$\left\|f^*-f_n^*\right\|_{\mathbf{X}}.$$

Let us first discuss what we can expect. In the last sentence of Proposition 4.4 we have already shown that the assumed mapping properties

$$\widehat{A}\in\mathcal{L}\big(\mathbf{Y}_\infty^{\mathcal{AB}},\mathbf{X}_\infty^{\mathcal{C}}\big),\quad H\in\mathcal{L}\big(\mathbf{X},\mathbf{X}_\infty^{\mathcal{C}}\big),\quad K\in\mathcal{L}\big(\mathbf{X},\mathbf{Y}_\infty^{\mathcal{AB}}\big)\tag{4.16}$$

together with the "smoothness" $g\in\mathbf{Y}_\infty^{\mathcal{AB}}$ of the right hand side of (4.7) imply

$$f^*\in\mathbf{X}_\infty^{\mathcal{C}}.\tag{4.17}$$

Thus, f^* can be approximated of order $O(c_n^{-1})$ by elements of $\mathbf{X}_n$ and in general we cannot expect that $\|f^*-f_n^*\|_{\mathbf{X}}$ converges faster too zero. Moreover, we have not assumed that the projections P_n are uniformly bounded in $\mathbf{X}$. For these reasons, an estimate of the type

$$\left\|f^*-f_n^*\right\|_{\mathbf{X}}\le c\,c_n^{-1}\,\|P_n\|_{\mathcal{L}(\mathbf{X})},\quad n\ge n_0\tag{4.18}$$

can be viewed as an optimal convergence result under the assumptions of Theorem 4.6. The aim of the present section is to demonstrate that indeed (4.18) holds true if the assumptions of Theorem 4.6 are satisfied with $\|L_n\|_{\mathcal{L}(\mathbf{Y})}\le c\,b_n/c_n$. Since the general situation is much more involved, we will restrict here on the simplest case

$$K_n=K\quad\text{and}\quad H_n=H,$$

only to show that the stability is the most important tool in the proof of error estimates. The verification of (4.18) in the case $K_n \neq K, H_n \neq H$ is given later.

Let us first investigate how good the operators P_n and $P_n\widehat{A}L_n$ which appear in (4.11) approximate I and $\widehat{A}$, respectively. More precisely, we are interested in estimates for $\|f - P_nf\|_{\mathbf{X}}$ and $\|\widehat{A}g - P_n\widehat{A}L_ng\|_{\mathbf{Y}}$, where $f \in \mathbf{X}_\infty^{\mathcal{C}}$ and $g \in \mathbf{Y}_\infty^{\mathcal{AB}}$.

If $f_n \in \mathbf{X}_n$ ($n \in \mathbb{N}$) is a best approximation of f in the norm of $\mathbf{X}$, then $\|f - f_n\|_{\mathbf{X}} \leq c_n^{-1}\|f\|_{\mathbf{X},\mathcal{C},\infty}$, where $\|\,.\,\|_{\mathbf{X},\mathcal{C},\infty}$ denotes the norm in $\mathbf{X}_\infty^{\mathcal{C}}$. (To distinguish the norms of approximation spaces based on $\mathbf{X}$ and $\mathbf{Y}$, we introduce the additional index $\mathbf{X}$ and $\mathbf{Y}$, respectively, in the notion of these norms.) Together with the decomposition

$$f - P_nf = f - f_n + P_n(f_n - f)$$

and the estimate $\|P_n\|_{\mathcal{L}(\mathbf{X})} \geq 1$ (which follows from $0 \neq P_n = P_n^2$; here we have to suppose that $n \geq n_0$ is large enough, such that $\mathbf{X}_n \neq \{0\}$) we obtain the following.

Lemma 4.8 $\|f - P_nf\|_{\mathbf{X}} \leq 2\,c_n^{-1}\|P_n\|_{\mathcal{L}(\mathbf{X})}\,\|f\|_{\mathbf{X},\mathcal{C},\infty}$ *for all* $f \in \mathbf{X}_\infty^{\mathcal{C}}$ *and all* $n \geq n_0$.

The first addend on the right hand side of $\widehat{A}g - P_n\widehat{A}L_ng = (\widehat{A}g - P_n\widehat{A}g) + P_n\widehat{A}(g - L_ng)$ can be estimated with the help of Lemma 4.8. Thus, for all $g \in \mathbf{Y}_\infty^{\mathcal{AB}}$ and all $n \geq n_0$, $\|\widehat{A}g - P_n\widehat{A}L_ng\|_{\mathbf{Y}} \leq c\,c_n^{-1}\|P_n\|_{\mathcal{L}(\mathbf{X})}\,\|g\|_{\mathbf{Y},\mathcal{AB},\infty} + \|P_n\|_{\mathcal{L}(\mathbf{X})}\|\widehat{A}(g - L_ng)\|_{\mathbf{X}}$ ($c \neq c(n,g)$). Here we have used the assumption $\widehat{A} \in \mathcal{L}(\mathbf{Y}_\infty^{\mathcal{AB}}, \mathbf{X}_\infty^{\mathcal{C}})$. Since also $\widehat{A} \in \mathcal{L}(\mathbf{Y}_1^{\mathcal{A}}, \mathbf{X})$, i.e.,

$$\|\widehat{A}\|_{\mathbf{Y}_n \to \mathbf{X}} \leq \operatorname{const} a_n\,, \quad n \in \mathbb{N} \tag{4.19}$$

(see Remark 1.22), $\widehat{A}$ is an operator as in Corollary 1.23, where the roles of $\mathbf{X}$ and $\mathbf{Y}$ have to be changed. Consequently,

$$\|\widehat{A}(g - L_ng)\|_{\mathbf{X}} \leq c\,a_n\|g - L_ng\|_{\mathbf{Y}} + c\,b_n^{-1}\|g\|_{\mathbf{Y},\mathcal{AB},\infty}$$

and it remains to mention that, clearly, Lemma 4.8 can be written down analogously for $g - L_ng$ instead of $f - P_nf$, where the sequence $\{c_n\}$ is replaced by $\{a_nb_n\}$. Let us summarize.

Lemma 4.9 $\|\widehat{A}g - P_n\widehat{A}L_ng\|_{\mathbf{X}} \leq c\left[c_n^{-1} + b_n^{-1}\|L_n\|_{\mathcal{L}(\mathbf{Y})}\right]\|P_n\|_{\mathcal{L}(\mathbf{X})}\,\|g\|_{\mathbf{Y},\mathcal{AB},\infty}$ *for all* $g \in \mathbf{Y}_\infty^{\mathcal{AB}}$ *and all* $n \geq n_0$*, where* n_0 *and* c *are independent of* n *and* g.

Using Lemmas 4.8, 4.9 and the mapping properties (4.16) it is easy to show that

$$I + P_nH + P_n\widehat{A}L_nK \longrightarrow I + H + \widehat{A}K \quad \text{in the norm of } \mathcal{L}(\mathbf{X})\,, \tag{4.20}$$

where we have the following estimate for the norm of $(I+H+\widehat{A}K)-(I+P_nH+P_n\widehat{A}L_nK)$,

$$\begin{aligned}\|(H + \widehat{A}K) - (P_nH + P_n\widehat{A}L_nK)\|_{\mathcal{L}(\mathbf{X})} &\leq \|H - P_nH\|_{\mathcal{L}(\mathbf{X})} + \|\widehat{A}K - P_n\widehat{A}L_nK\|_{\mathcal{L}(\mathbf{X})} \\ &\leq c\left[c_n^{-1} + b_n^{-1}\|L_n\|_{\mathcal{L}(\mathbf{Y})}\right]\|P_n\|_{\mathcal{L}(\mathbf{X})}\,, \end{aligned} \tag{4.21}$$

which holds for all sufficiently large $n \geq n_0$. (Remember that we have supposed that the right hand side of (4.21) converges to zero.) It is well known that inversion is a continuous operation on the open set of all invertible operators of the Banach algebra

$\mathcal{L}(\mathbf{X})$. Thus, from the assumed invertibility of $I + H + \widehat{A}K$ and (4.20) it follows that, in the case $K_n = K, H_n = H$, the approximation method (4.11) is even stable in $\mathbf{X}$, i.e., for sufficiently large $n \geq n_0$ the operators on the left hand side of (4.20) are invertible in $\mathcal{L}(\mathbf{X})$, where

$$\sup_{n \geq n_0} \left\| (I + P_n H + P_n \widehat{A} L_n K)^{-1} \right\|_{\mathcal{L}(\mathbf{X})} < \infty \,. \tag{4.22}$$

This implies that, for $n \geq n_0$, (4.11) (with $K_n = K$ and $H_n = H$) possesses a unique solution f_n^* in $\mathbf{X}$. Clearly, $f_n^* = P_n\big[\widehat{A}L_n(g - Kf_n^*) - Hf_n^*\big]$ belongs to $\mathbf{X}_n$.

(4.22) is a stronger kind of stability than that considered in Theorem 4.6. This is the reason why the proof of (4.18) becomes easier if H and K are not replaced by approximative operators. Indeed, in this case we can write

$$f^* - f_n^* = (I + P_n H + P_n \widehat{A} L_n K)^{-1} \big[(I + P_n H + P_n \widehat{A} L_n K) f^* - P_n \widehat{A} L_n g \big]$$

for all $n \geq n_0$. Taking $f^* = \widehat{A}g - (H + \widehat{A}K)f^*$ and (4.22) into account, we obtain

$$\left\| f^* - f_n^* \right\|_{\mathbf{X}} \leq c \left(\left\| \widehat{A}g - P_n \widehat{A} L_n g \right\|_{\mathbf{X}} + \left\| \big[(P_n H + P_n \widehat{A} L_n K) - (H + \widehat{A}K) \big] f^* \right\|_{\mathbf{X}} \right).$$

If $\|L_n\|_{\mathcal{L}(\mathbf{Y})} \leq c\, b_n / c_n$ then, together with Lemma 4.9 and estimate (4.21), this yields (4.18) in case $H_n = H$, $K_n = K$. More precisely, in this case we have the following estimate in which the constants c and n_0 are independent of g and n,

$$\left\| f^* - f_n^* \right\|_{\mathbf{X}} \leq c \left(c_n^{-1} + b_n^{-1} \, \|L_n\|_{\mathcal{L}(\mathbf{Y})} \right) \|P_n\|_{\mathcal{L}(\mathbf{X})} \left(\|g\|_{\mathbf{Y}, \mathcal{AB}, \infty} + \|f^*\|_{\mathbf{X}} \right), \quad n \geq n_0 \,. \tag{4.23}$$

If $(H_n, K_n) \neq (H, K)$, then (4.22) with H and K replaced by H_n and K_n, respectively, is not true in general. In view of Theorem 4.6, we also have to replace $\| \,.\, \|_{\mathcal{L}(\mathbf{X})}$ by $\| \,.\, \|_{\mathcal{L}(\mathbf{X}_n)}$. Nevertheless, (4.23) remains true.

Theorem 4.10 *Let the assumptions of Theorem 4.6 be satisfied and let $g \in \mathbf{Y}_\infty^{\mathcal{AB}}$. If $\big(I + H + \widehat{A}K\big) f = 0$ has only the trivial solution in $\mathbf{X}$, then (4.9) and (4.11) (with sufficiently large n) have uniquely determined solutions $f^* \in \mathbf{X}$ and $f_n^* \in \mathbf{X}_n$, respectively, and*

$$\left\| f^* - f_n^* \right\|_{\mathbf{X}} \leq c \left(c_n^{-1} + b_n^{-1} \|L_n\|_{\mathcal{L}(\mathbf{Y})} \right) \|P_n\|_{\mathcal{L}(\mathbf{X})} \, \|g\|_{\mathbf{Y}, \mathcal{AB}, \infty} \quad \text{for all } \; n \geq n_0 \,, \tag{4.24}$$

where n_0 and c are independent of g and n.

Remark 4.11 *In view of (4.17), f^* from Theorem 4.10 belongs to $\mathbf{X}_\infty^{\mathcal{C}}$. Moreover, the assumption* $\ker\big(I + H + \widehat{A}K\big) = \{0\}$ *is equivalent to the uniqueness in $\mathbf{X}_\infty^{\mathcal{C}}$ of the solution of $\big(I + H + \widehat{A}K\big) f = 0$ (since, by (4.16),* $\ker\big(I + H + \widehat{A}K\big) \subseteq \mathbf{X}_\infty^{\mathcal{C}}$*).*

4.3 Proofs

4.3.1 Proof of Theorem 4.6

The following result is well known. We give its proof for the sake of completeness.

Lemma 4.12 *Let $T \in \mathcal{L}(\mathbf{X})$ and $T_n \in \mathcal{L}(\mathbf{X}_n)$ such that*

$$\lim_{n\to\infty} \|T - T_n\|_{\mathbf{X}_n\to\mathbf{X}} = 0\,. \tag{4.25}$$

If $I + T$ is invertible in $\mathcal{L}(\mathbf{X})$, then $A_n = I + T_n$ is stable in the sense of (4.15).

Proof. For $f_n \in \mathbf{X}_n$ we have

$$\begin{aligned}\|(I+T_n)f_n\|_{\mathbf{X}} &\geq \|(I+T)f_n\|_{\mathbf{X}} - \|(T_n - T)f_n\|_{\mathbf{X}}\\ &\geq \frac{\|f_n\|_{\mathbf{X}}}{\|(I+T)^{-1}\|_{\mathcal{L}(\mathbf{X})}} - \|T_n - T\|_{\mathbf{X}_n\to\mathbf{X}}\|f_n\|_{\mathbf{X}}\,.\end{aligned}$$

Thus, $\|(I+T_n)f_n\|_{\mathbf{X}} \geq c\,\|f_n\|_{\mathbf{X}}$ for $n \geq n_0$. This estimate implies that $I + T_n$ is invertible in $\mathcal{L}(\mathbf{X}_n)$ for all $n \geq n_0$ (since $\dim \mathbf{X}_n < \infty$) and that the norms of the inverses are uniformly bounded (take $f_n = (I+T_n)^{-1}g_n$, $g_n \in \mathbf{X}_n$). ■

Proof of Theorem 4.6. We show that the assumptions of Lemma 4.12 are satisfied for

$$T = H + \widehat{A}K \quad \text{and} \quad T_n = P_nH_n + P_n\widehat{A}L_nK_n\,.$$

First we recall that, by (4.10), $T \in \mathcal{K}(\mathbf{X})$. Consequently, $I+T$ is invertible in $\mathcal{L}(\mathbf{X})$, since we have supposed $\ker(I+T) = \{0\}$. Moreover, it is clear that $T_n \in \mathcal{L}(\mathbf{X}_n)$. To prove (4.25) we use the estimate

$$\begin{aligned}\|T - T_n\|_{\mathbf{X}_n\to\mathbf{X}} \leq\; &\big\|(H + \widehat{A}K) - (P_nH + P_n\widehat{A}L_nK)\big\|_{\mathcal{L}(\mathbf{X})}\\ &+\big\|P_n(H - H_n)\big\|_{\mathbf{X}_n\to\mathbf{X}} + \|P_n\|_{\mathcal{L}(\mathbf{X})}\big\|\widehat{A}L_n(K - K_n)\big\|_{\mathbf{X}_n\to\mathbf{X}}\,.\end{aligned}$$

In Section 4.2 it is proved that, for $n \geq n_0$,

$$\big\|(H + \widehat{A}K) - (P_nH + P_n\widehat{A}L_nK)\big\|_{\mathcal{L}(\mathbf{X})} \leq c\left(c_n^{-1} + b_n^{-1}\,\|L_n\|_{\mathcal{L}(\mathbf{Y})}\right)\|P_n\|_{\mathcal{L}(\mathbf{X})}$$

(see (4.21)). Furthermore, for every $f_n \in \mathbf{X}_n$ and every $h_n \in \ker P_n$,

$$\big\|P_n(H - H_n)f_n\big\|_{\mathbf{X}} = \big\|P_n[(H-H_n)f_n - h_n]\big\|_{\mathbf{X}} \leq \|P_n\|_{\mathcal{L}(\mathbf{X})}\,\big\|(H-H_n)f_n - h_n\big\|_{\mathbf{X}}\,.$$

If we take the infimum over all $h_n \in \ker P_n$ and remember the assumption (ii) of Theorem 4.6, then we obtain $\|P_n(H - H_n)f_n\|_{\mathbf{X}} \leq c\,c_n^{-1}\|P_n\|_{\mathcal{L}(\mathbf{X})}\|f_n\|_{\mathbf{X}}$. Thus,

$$\big\|P_n(H - H_n)\big\|_{\mathbf{X}_n\to\mathbf{X}} \leq c\,c_n^{-1}\|P_n\|_{\mathcal{L}(\mathbf{X})}\,. \tag{4.26}$$

In view of (4.19) we have

$$\big\|\widehat{A}L_n(K - K_n)\big\|_{\mathbf{X}_n\to\mathbf{X}} \leq c\,a_n\big\|L_n(K - K_n)\big\|_{\mathbf{X}_n\to\mathbf{Y}}$$

Similar to the proof of (4.26) we obtain

$$\big\|L_n(K - K_n)\big\|_{\mathbf{X}_n\to\mathbf{Y}} \leq c\,a_n^{-1}\big(b_n^{-1} + c_n^{-1}\,\|L_n\|_{\mathcal{L}(\mathbf{Y})}^{-1}\big)\|L_n\|_{\mathcal{L}(\mathbf{Y})}\,,$$

where we took assumption (i) of Theorem 4.6 into account. Consequently,

$$\|T - T_n\|_{\mathbf{X}_n\to\mathbf{X}} \leq c\left(c_n^{-1} + b_n^{-1}\|L_n\|_{\mathcal{L}(\mathbf{Y})}\right)\|P_n\|_{\mathcal{L}(\mathbf{X})}$$

and from assumption (iii) of Theorem 4.6 it follows $\|T - T_n\|_{\mathbf{X}_n\to\mathbf{X}} \to 0$. ■

4.3.2 Proof of Theorem 4.10

In the proof of Theorem 4.6 we have already shown that there exist uniquely determined solutions f^* and f_n^* $(n \ge n_0)$ of (4.7) and (4.11), respectively, and that

$$\left\| P_n(H_n - H) + P_n \widehat{A} L_n (K_n - K) \right\|_{\mathbf{X}_n \to \mathbf{X}} \le c \left(c_n + b_n^{-1} \|L_n\|_{\mathcal{L}(\mathbf{Y})} \right) \|P_n\|_{\mathcal{L}(\mathbf{X})} \tag{4.27}$$

for all $n \in \mathbb{N}$. Furthermore, we know from Section 4.2 that, for $n \ge n_0$, the equations

$$\left(I + P_n H + P_n \widehat{A} L_n K \right) f_n = P_n \widehat{A} L_n g$$

have uniquely determined solutions $f_n^{**} \in \mathbf{X}_n$, where

$$\left\| f^* - f_n^{**} \right\|_{\mathbf{X}} \le c \left(c_n^{-1} + b_n^{-1} \|L_n\|_{\mathcal{L}(\mathbf{Y})} \right) \|P_n\|_{\mathcal{L}(\mathbf{X})} \left(\|g\|_{\mathbf{Y}, \mathcal{AB}, \infty} + \|f^*\|_{\mathbf{X}} \right) \tag{4.28}$$

(see (4.23)). Now we estimate $\left\| f_n^{**} - f_n^* \right\|_{\mathbf{X}}$. For this aim, we write

$$f_n^{**} - f_n^* = \left(I + P_n H_n + P_n \widehat{A} L_n K_n \right)^{-1} \left[\left(I + P_n H_n + P_n \widehat{A} L_n K_n \right) f_n^{**} - P_n \widehat{A} L_n g \right].$$

If we take into account that $\left(I + P_n H_n + P_n \widehat{A} L_n K_n \right)^{-1}$, $n \ge n_0$, is uniformly bounded in $\mathcal{L}(\mathbf{X}_n)$ (see Theorem 4.6) and that $f_n^{**} = P_n \widehat{A} L_n g - P_n H f_n^{**} - P_n \widehat{A} L_n K f_n^{**}$, then we obtain

$$\left\| f_n^{**} - f_n^* \right\|_{\mathbf{X}} \le c \left\| \left[P_n(H_n - H) + P_n \widehat{A} L_n (K_n - K) \right] f_n^{**} \right\|_{\mathbf{X}}.$$

Together with (4.27) this yields

$$\left\| f_n^{**} - f_n^* \right\|_{\mathbf{X}} \le c \left(c_n^{-1} + b_n^{-1} \|L_n\|_{\mathcal{L}(\mathbf{Y})} \right) \|P_n\|_{\mathcal{L}(\mathbf{X})} \|f_n^{**}\|_{\mathbf{X}}. \tag{4.29}$$

If we write $f^* = (I + H + \widehat{A} K)^{-1} \widehat{A} g$, then it becomes clear that

$$\|f^*\|_{\mathbf{X}} \le c \, \|g\|_{\mathbf{Y}, \mathcal{AB}, \infty}. \tag{4.30}$$

Now we obtain from (4.28) that also

$$\|f_n^{**}\|_{\mathbf{X}} \le c \, \|g\|_{\mathbf{Y}, \mathcal{AB}, \infty}. \tag{4.31}$$

Estimates (4.28)–(4.31) yield the assertion (4.24). ■

4.4 Notes and comments

The results of Chapter 4 are new. Only in the case of general spectral methods, i.e., under the additional assumptions $H = 0$, $A(\mathbf{X}_n) \subseteq \mathbf{Y}_n$ and $\widehat{A}(\mathbf{Y}_n) \subseteq \mathbf{X}_n$ (compare Remark 4.3), stability and convergence results similar to Theorems 4.6 and 4.10 are given in [L4]. But in [L4] the condition (i) of Theorem 4.6 is replaced by a much stronger condition. The ideas of the proofs of Theorems 4.6 and 4.10 are based on the investigations of the weighted uniform convergence of spectral methods for Cauchy singular integral equations; see [JL1, JL2, L1]. Let us shortly explain how the considerations of Chapter 4 have to be modified in the case of such a spectral method.

At the beginning of Section 2.6 we have already mentioned that spectral methods for equation (2.45) are of the following form,

$$f_n \in \mathbf{X}_n := \sigma \Pi_{n+\varkappa} : \ (A + L_n K_n) f_n = L_n g \,, \tag{4.32}$$

where $\sigma = v^{k,l}\sigma_0$ and L_n is a projection onto $\mathbf{Y}_n = A(\mathbf{X}_n)$. Here we suppose that (2.45) possesses a solution f in $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ ($\alpha = \alpha_0 + k$, $\beta = \beta_0 + l$) and we assume that b satisfies the additional assumptions given in Theorem 2.11 if $\varkappa \neq 0$. Thus, in the case $\varkappa \leq 0$, g satisfies (2.29) and f is uniquely determined (see Remark 2.14). In this case one can show that

$$\mathbf{Y}_n = \Pi_n \cap \big(b\,\mu\,\Pi_{-\varkappa}\big)^\perp$$

and that (4.32) is equivalent to

$$f_n \in \mathbf{X}_n : \ (I + B_{k,l} L_n K_n) f_n = B_{k,l} L_n g \tag{4.33}$$

(since $AB_{k,l}$ is a projection from $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ onto $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap \big(b\,\mu\,\Pi_{-\varkappa}\big)^\perp$). If $\varkappa > 0$, then the operator $B_{k,l}A$ is a projection from $v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ onto $B_{k,l}\big(v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})\big) = v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S}) \cap \big(b\,\Pi_{\varkappa}\big)^\perp$. Hence, (4.32) together with the additional condition $f_n \in \big(b\,\Pi_{\varkappa}\big)^\perp$ is equivalent to (4.33) and the solutions f_n of (4.33) are approximations of that solution $f \in v^{\alpha^+,\beta^+}\mathbf{H}_{\mathrm{loc}}(\mathcal{S})$ of (2.45) which satisfies $f \in \big(b\,\Pi_{\varkappa}\big)^\perp$ or, equivalently,

$$(I + B_{k,l}K) f = B_{k,l}\, g \,. \tag{4.34}$$

The stability and convergence of the spectral method (4.33) for equation (4.34) can be investigated in the same way as in the previous sections, where we take $\widehat{A} = B_{k,l}$, $H = 0$, $\mathbf{X} = \mathbf{Y} = \mathbf{C}_u$ with u of the form (2.19) satisfying (2.52), $\mathcal{A} = \{\log_2(n+1)\}$, and $b_n \sim (n+1)^\gamma \ln^\delta n$ (supposed that $g \in \mathbf{C}_u^{\gamma,\delta+1}$ for some $\gamma > 0$). Only P_n has to be replaced by I. In this way known results from [JL1] and [JL2] can be generalized to the case of power weights u. We omit the details since spectral methods are not the subject of this paper.

Chapter 5

Numerical analysis for Cauchy singular integral equations

5.1 The regularized equation

In all of what follows we consider a Cauchy singular integral equation of the type

$$(A+K)\,\phi = g \qquad (g \text{ is given, } \phi \text{ is looked for})\,, \tag{5.1}$$

where, as in Section 2.3, A denotes the operator

$$A = aI + SbI$$

with real-valued and Hölder continuous coefficient functions a and b satisfying

$$r(x) = \sqrt{a^2(x)+b^2(x)} > 0 \quad \text{for all} \quad x \in [-1,1]\,.$$

The operator K is a weakly singular integral operator,

$$(K\phi)(x) = \frac{1}{\pi}\int_{-1}^{1} k(x,t)\,\phi(t)\,dt\,,$$

to which the results of Chapter 3 can be applied. More precisely, we suppose that there is a power weight

$$u(x) = \prod_{j=0}^{N+1} |x-x_j|^{\tau_j} \quad \text{with} \quad \begin{array}{l} -1 = x_0 < x_1 < \ldots < x_{N+1} = 1\,, \\ 0 < \tau_j < 1 \ \text{ for all } \ j = 1,\ldots,N, \\ 0 \le \tau_0 < 1\,,\ \ 0 \le \tau_{N+1} < 1\,, \\ \tau_0 > 0 \ \text{ if } \ b(-1) = 0\,,\ \ \tau_{N+1} > 0 \ \text{ if } \ b(1) = 0 \end{array} \tag{5.2}$$

($N \in \mathbb{N}_0$) and a Jacobi weight

$$w(x) = v^{\mu,\nu}(x) \quad \text{with} \quad -1 < \mu < 1 - \tau_{N+1} \ \text{ and } \ -1 < \nu < 1 - \tau_0 \tag{5.3}$$

such that $k(x,t)$ can be decomposed into a sum $k(x,t) = k_1(x,t) + k_2(x,t)$ the addends of which satisfy the following assumptions:

$h_1(x,t) = u(x)\,k_1(x,t)\,w(t)$ is Hölder continuous on $[-1,1]^2$ and

$h_1(x_j,\,.\,) \equiv 0$ for all j with $\tau_j > 0$, $h_1(\,.\,,1) \equiv 0$ if $\mu \neq 0$, $h_1(\,.\,,-1) \equiv 0$ if $\nu \neq 0$,

$h_2(x,t) = (t-x)\,k_2(x,t)$ is Hölder continuous on $[-1,1]^2$ and $h_2(x,x) = 0$ on $[-1,1]$.

Hölder continuity is meant in the sense that the functions are defined on $\mathrm{supp}_* u \times (-1,1)$ and $[-1,1]^2 \setminus \{x=t\}$, respectively, and can be extended to Hölder continuous functions on $[-1,1]^2$. We further suppose that the right hand side g of (5.1) belongs to $\mathbf{C}_u$, where

$$g\,u \text{ is Hölder continuous on } [-1,1] \text{ and } (g\,u)(x_j) = 0 \text{ for all } j \text{ with } \tau_j > 0\,.$$

Proposition 5.1 *The above conditions on g and k are equivalent to the conditions*

$$h_1(x,t) = u(x)\,k_1(x,t)\,w(t) \in \mathbf{C}\big([-1,1]^2\big),\ \ h_2(x,t) = (t-x)\,k_2(x,t) \in \mathbf{C}\big([-1,1]^2\big) \tag{5.4}$$

(u and w satisfying (5.2) and (5.3), respectively),

$$h_2(x,x) = 0 \quad \textit{for all } \ x \in [-1,1]\,, \tag{5.5}$$

and to the existence of constants $\gamma > 0$ and $\delta \in \mathbb{R}$ such that

$$g \in \mathbf{C}_u^{\gamma,\delta}\,, \tag{5.6}$$

$$\sup_{t\in[-1,1]} \big\|u^{-1}(\,.\,)\,h_1(\,.\,,t)\big\|_{u,\gamma,\delta} < \infty\,, \quad \sup_{x\in[-1,1]} \big\|h_1(x,\,.\,)\,w^{-1}(\,.\,)\big\|_{w,\gamma,\delta} < \infty \tag{5.7}$$

$$\sup_{t\in[-1,1]} \big\|h_2(\,.\,,t)\big\|_{\gamma,\delta+1} < \infty\,, \quad \sup_{x\in[-1,1]} \big\|h_2(x,\,.\,)\big\|_{\gamma,\delta+1} < \infty\,. \tag{5.8}$$

Moreover, the suprema in (5.7) and (5.8) can be replaced by essential suprema, i.e., this does not weaken the assumptions.

Remark 5.2 *If w is unbounded in -1 or 1, then we have to modify the definition of $\mathbf{C}_u^{\gamma,\delta}$ (which was only given for continuous weights u in Section 1.2) to obtain $\mathbf{C}_w^{\gamma,\delta}$ and its norm $\|\,.\,\|_{w,\gamma,\delta}$. Namely, $f : (-1,1) \to \mathbb{C}$ belongs to $\mathbf{C}_w^{\gamma,\delta}$ iff $f \in \mathbf{C}_w$ (i.e., fw possesses a continuous extension on $[-1,1]$) and $\|f\|_{w,\gamma,\delta} := \sup_{n=0,1,\dots} E_n^w(f)(n+1)^\gamma \ln^\delta(n+2) < \infty$, where $E_n^w(f) = \inf\big\{\|fw - p_n\,w\| : p_n \in \Pi_n \cap \mathbf{C}_w\big\}$. One can show that $\|f\|_{w,\gamma,\delta} \sim \big\|p^{-1}f\big\|_{wp,\gamma,\delta}$ for all $f \in \mathbf{C}_w$, where $p = v^{m(\mu),m(\nu)}$, $m(\mu) = 0$ if $\mu \ge 0$, $m(\mu) = 1$ if $\mu < 0$. (See the proof of Proposition 5.1.)*

Before we can study approximation methods for (5.1), we must ask for the existence and uniqueness of the solution ϕ. Of course, the answer depends on the space in which we look for ϕ. We will consider the linear space

$$\mathbf{H}_u := \big\{f \in \mathbf{C}_u : fu \text{ is Hölder continuous on } [-1,1]\big\}\,.$$

The left hand side $(A+K)\phi$ of (5.1) is well-defined for $\phi \in \mathbf{H}_u$ because of the following result.

Proposition 5.3 *If the assumptions (5.2)–(5.8) are satisfied, then*

$$A \in \mathbb{L}\big(\mathbf{H}_u,\, \mathbf{H}_{\mathrm{loc}}\big(\{x_j\}_{j=0}^{N+1}\big)\big) \quad \textit{and} \quad K \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}_u^{\gamma,\delta}\big)\,. \tag{5.9}$$

Clearly, the weight u which appears in the above assumptions is not uniquely determined. Now we simply suppose that u is chosen in such a way that

$$\text{(5.1) possesses a uniquely determined solution } \phi \in \mathbf{H}_u \; . \tag{5.10}$$

Often (5.10) can be assumed because of considerations on the underlying practical problem which is modelled by (5.1). Thereby, the following result can be useful in order to find the right weight u.

Proposition 5.4 *Let the assumptions (5.2)–(5.8) be satisfied and let α_0 and β_0 be defined by (2.55). Then, every solution $\phi \in \mathbf{H}_u$ of (5.1) has the additional properties*

$$(\phi u)(x_j) = 0 \;\; \textit{for all} \;\; j = 1, \ldots, N \quad \textit{and} \quad \left\{ \begin{array}{ll} (\phi u)(-1) = 0 & \textit{if } \tau_0 \neq -\beta_0 \,, \\ (\phi u)(1) = 0 & \textit{if } \tau_{N+1} \neq -\alpha_0 \,. \end{array} \right.$$

If, in addition, $k_1(x,t)w(t)$ is Hölder continuous on $[-1,-1+\varepsilon]\times[-1,1]$ ($[1-\varepsilon,1]\times[-1,1]$), $k_2 = 0$, and g is Hölder continuous on $[-1,-1+\varepsilon]$ ($[1-\varepsilon,1]$), then every solution $\phi \in \mathbf{H}_u$ of (5.1) is also Hölder continuous on $[-1,-1+\varepsilon]$ ($[1-\varepsilon,1]$) and $\phi(-1) = 0$ ($\phi(1) = 0$).

If one has no idea whether (5.10) is satisfied or not, then it is probably the best to determine u as in the following proposition (if possible) in which the case $K = 0$ is considered, and then to hope that, for the concrete $K \neq 0$ under consideration, the dimension of the set of solutions $\phi \in \mathbf{H}_u$ of (5.1) does not differ from that of $A\phi = g$.

Proposition 5.5 *Let α_0, β_0, and $\varkappa_0$ be defined by (2.55) and (2.25), respectively. Suppose that $\varkappa_0 \in \{0,1,2\}$ and that $K = 0$. Then, (5.10) is satisfied if u is chosen in such a way that $g \in \mathbf{C}_u^{\gamma,\delta}$ for certain $\gamma > 0$, $\delta \in \mathbb{R}$, and*

$$\left\{ \begin{array}{ll} \tau_0 \geq -\beta_0 \;\; \textit{and} \;\; \tau_{N+1} \geq -\alpha_0 & \textit{if} \;\; \varkappa_0 = 0 \,, \\ \big(\tau_0 \geq -\beta_0 \;\; \textit{and} \;\; \tau_{N+1} < -\alpha_0\big) \;\; \textit{or} \;\; \big(\tau_0 < -\beta_0 \;\; \textit{and} \;\; \tau_{N+1} \geq -\alpha_0\big) & \textit{if} \;\; \varkappa_0 = 1 \,, \\ \tau_0 < -\beta_0 \;\; \textit{and} \;\; \tau_{N+1} < -\alpha_0 & \textit{if} \;\; \varkappa_0 = 2 \,. \end{array} \right.$$

Now we suppose that we have chosen the weights u and w such that the assumptions (5.2)–(5.8) and the condition (5.10) are satisfied. Motivated by the considerations of Section 2.6, we want to regularize equation (5.1) with the help of the left $\mathbf{C}_u$–regularizer $\widehat{A}$ from Theorem 2.21. In the approximation method studied in the next section the weights $\sigma_{-\alpha,-\beta}$ and $\sigma_{\alpha,\beta} = \sigma_{-\alpha,-\beta}^{-1} \cos^2(\ell_{\alpha,\beta})$ from Theorem 2.21 play an important role. Particularly, we will see that it is advantageous to write the solution $\phi \in \mathbf{H}_u$ of (5.1) in the form

$$\phi = \sigma_{\alpha,\beta} \, f \, . \tag{5.11}$$

Since $\sigma_{\alpha,\beta}$ is the product of $v^{\alpha,\beta}$ and a positive Hölder continuous function, condition (5.10) is equivalent to the existence and uniqueness of a solution

$$f \in \mathbf{H}_{u(\alpha,\beta)} \, , \quad u(\alpha,\beta) = v^{\alpha,\beta} u \, ,$$

of the equation

$$(\mathcal{A} + \mathcal{K}) f = g \, , \quad \text{where} \quad \mathcal{A} := A\sigma_{\alpha,\beta} I \, , \;\; \mathcal{K} := K\sigma_{\alpha,\beta} I \, . \tag{5.12}$$

Instead of the regularizer $\widehat{A}$ from Theorem 2.21, we now consider the left regularizer

$$\widehat{\mathcal{A}} := \sigma_{\alpha,\beta}^{-1}\widehat{A} = \frac{1}{r^2\cos^2(\ell_{\alpha,\beta})}(aI - bS)\sigma_{-\alpha,-\beta}I = \frac{\widetilde{a}}{r^2\sigma_{\alpha,\beta}}I - \frac{b}{r^2\cos^2(\ell_{\alpha,\beta})}A_{-\alpha,-\beta}$$

($\widetilde{a} = a - b\tan(\ell_{\alpha,\beta})$) of $\mathcal{A}$. From (5.9), (2.44) (applied to $A_{-\alpha,-\beta}$), and Proposition 1.12 it follows

$$\mathcal{K} \in \mathcal{L}\big(\mathbf{C}_{u(\alpha,\beta)}, \mathbf{C}_u^{\gamma,\delta}\big) \quad \text{and} \quad \widehat{\mathcal{A}} \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_{u(\alpha,\beta)}^{s,0}\big) \tag{5.13}$$

for some $s > 0$. (Take Corollary 1.18, (v), Proposition 1.19, (i), and (2.63) into account to investigate the part $[\widetilde{a}/(r^2\sigma_{\alpha,\beta})]\,I$ of $\widehat{\mathcal{A}}$.) Together with Theorem 2.21 (and the Hölder continuity of $[r\cos(\ell_{\alpha,\beta})]^{-2}$) we obtain the following.

Proposition 5.6 *Let the assumptions (5.2)–(5.8) be satisfied, take α, β, $\widehat{A}$, H from Theorem 2.21 and let $\widehat{\mathcal{A}}$ and $\mathcal{K}$ be defined as above. If $f \in \mathbf{H}_{u(\alpha,\beta)}$ is a solution of (5.12), then f is also a solution of the regularized equation*

$$(I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K})f = \widehat{\mathcal{A}}\,g\,, \tag{5.14}$$

where $\mathcal{H} = \sigma_{\alpha,\beta}^{-1}H\sigma_{\alpha,\beta}I$, i.e., $(\mathcal{H}f)(x)$, $x \in (-1,1)\setminus\{x_j\}_{j=1}^N$, is given by

$$\frac{\pi^{-1}}{[r\cos(\ell_{\alpha,\beta})]^2(x)}\int_{-1}^{1}\left[\frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\,b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\,b(x)}{t-x} - \frac{\pi\,\widetilde{k}\,\widetilde{l}\,b(x)\,b(t)}{\int_{-1}^{1}\sigma_{\alpha,\beta}(\tau)\,d\tau}\right]\phi(t)\,dt$$

with $\phi = \sigma_{\alpha,\beta}f$ and $\widetilde{a}$, $\widetilde{k}$, $\widetilde{l}$ from (2.59). Moreover, there exists some $s > 0$ such that

$$\mathcal{H} \in \mathcal{L}\big(\mathbf{C}_{u(\alpha,\beta)}, \mathbf{C}_{u(\alpha,\beta)}^{s,0}\big)\,. \tag{5.15}$$

In view of (5.13), (5.15), and assertion (i) of Theorem 1.11, $\mathcal{H} + \widehat{\mathcal{A}}\mathcal{K} \in \mathcal{K}(\mathbf{C}_{u(\alpha,\beta)})$ and $\widehat{\mathcal{A}}g \in \mathbf{C}_{u(\alpha,\beta)}$. Thus, it is reasonable to consider (5.14) as an operator equation in the space $\mathbf{C}_{u(\alpha,\beta)}$. From now on we will suppose (in addition to the assumptions (5.2)–(5.8)) that

$$\text{(5.14) possesses a uniquely determined solution } f \in \mathbf{C}_{u(\alpha,\beta)}\,. \tag{5.16}$$

In view of Proposition 5.6, this ensures that a solution $\phi \in \mathbf{H}_u$ of (5.1), supposed that it exists, is uniquely determined by (5.11). Of course, (5.16) is different from the assumption (5.10). But, in view of the following fact, there is some reasonable hope that, if one considers a certain concrete equation, also (5.16) holds true if (5.10) is satisfied. (Of course, we do not state that this is true in general.)

Remark 5.7 *The mapping properties (5.13) and (5.15) together with assertion (i) of Proposition 1.19 show that, under the assumptions (5.2)–(5.8), a $\mathbf{C}_{u(\alpha,\beta)}$-solution $f = \widehat{\mathcal{A}}(g - \mathcal{K}f) - \mathcal{H}f$ of (5.14) automatically belongs to $\mathbf{H}_{u(\alpha,\beta)}$.*

5.2 The quadrature method for the regularized equation

To solve the equation (5.14) numerically, we will consider an approximation method of the type (4.11) to which the results of Chapter 4 can be applied. For the practical realization of such a method it is important that, for given g, $(\widehat{\mathcal{A}}\, L_n g)(x)$ can be computed in that points x which are necessary to determine $P_n \widehat{\mathcal{A}}\, L_n g$. Later (in Section 5.4) we will see that it is possible (with acceptable computational effort) to determine the values of $\widehat{\mathcal{A}}\, p_n$, $p_n \in \Pi_{2n-1}$, in the zeros of the orthogonal polynomial $p_n^{\alpha,\beta}$ with respect to the weight $\sigma_{\alpha,\beta}$ if the values of p_n in the zeros of $p_{n(\alpha,\beta)}^{-\alpha,-\beta}$,

$$n(\alpha,\beta) := n + 1 - \widetilde{k} - \widetilde{l} \quad \big(\widetilde{k},\, \widetilde{l} \text{ from } (2.59)\big)\,, \tag{5.17}$$

are given. For this reason we will take certain modified Lagrangian interpolation operators L_n and P_n with respect to the zeros of $p_{n(\alpha,\beta)}^{-\alpha,-\beta}$ and $p_n^{\alpha,\beta}$, respectively. Let us turn to the details. First we have to give the precise definition of the orthogonal polynomials $p_n^{\alpha,\beta}$.

Definition 5.8 *Let $\alpha, \beta \in (-1,1) \setminus \{0\}$. By $p_n^{\alpha,\beta}$ ($n \in \mathbb{N}_0$) we denote the unique polynomial of degree n which satisfies*

$$\int_{-1}^{1} p(x)\, p_n^{\alpha,\beta}(x)\, \sigma_{\alpha,\beta}(x)\, dx = 0 \quad \textit{for all } p \in \Pi_n \quad \textit{and}$$

$$p_n^{\alpha,\beta}(x) = c_n^{\alpha,\beta} x^n + \ldots \quad \textit{with } c_n^{\alpha,\beta} > 0 \textit{ such that } \frac{1}{\pi}\int_{-1}^{1} \big[p_n^{\alpha,\beta}(x)\big]^2 \sigma_{\alpha,\beta}(x)\, dx = 1\,.$$

For $n < 0$ we set $p_n^{\alpha,\beta} = 0$.

The proof of the existence and uniqueness of $p_n^{\alpha,\beta}$ can be found in any book on orthogonal polynomials, for example in [F]. The polynomials $p_n^{\alpha,\beta}$ can be determined if the coefficients of a certain recurrence formula are given. In a later section about computational aspects we will present this recurrence formula and show how the recurrence coefficients can be determined numerically. Moreover, we will demonstrate there that also the following knots and zeros of the Gaussian quadrature rule with respect to $\sigma_{\alpha,\beta}$ can be computed if the recurrence coefficients of $\big\{p_m^{\alpha,\beta}\big\}_{m=0}^{n}$ are given.

Definition 5.9 *Let $\alpha, \beta \in (-1,1) \setminus \{0\}$ and $n \in \mathbb{N}$. The zeros of $p_n^{\alpha,\beta}$ (which are all simple and lie in $(-1,1)$) are denoted by $x_{n,j}^{\alpha,\beta}$, where*

$$-1 < x_{n,1}^{\alpha,\beta} < x_{n,2}^{\alpha,\beta} < \ldots < x_{n,n}^{\alpha,\beta} < 1\,.$$

The Lagrangian interpolation operator $L_n^{\alpha,\beta} : \mathbf{C}(D) \to \Pi_n$ (D: some closed set which contains $\big\{x_{n,j}^{\alpha,\beta}\big\}_{j=1}^{n}$) with respect to these zeros is defined by

$$(L_n^{\alpha,\beta} f)(x) = \sum_{j=1}^{n} f\big(x_{n,j}^{\alpha,\beta}\big)\, l_{n,j}^{\alpha,\beta}(x)\,, \quad l_{n,j}^{\alpha,\beta}(x) := \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{x - x_{n,i}^{\alpha,\beta}}{x_{n,j}^{\alpha,\beta} - x_{n,i}^{\alpha,\beta}}$$

(i.e., $L_n^{\alpha,\beta} f$ is that polynomial of degree $< n$ which satisfies $\big(L_n^{\alpha,\beta} f\big)\big(x_{n,j}^{\alpha,\beta}\big) = f\big(x_{n,j}^{\alpha,\beta}\big)$, $j = 1, \ldots, n$). The functional $Q_n^{\alpha,\beta} : \mathbf{C}(D) \to \mathbb{C}$ defined by

$$Q_n^{\alpha,\beta} f := \frac{1}{\pi} \int_{-1}^{1} (L_n^{\alpha,\beta} f)(x)\, \sigma_{\alpha,\beta}(x)\, dx = \sum_{j=1}^{n} \lambda_{n,j}^{\alpha,\beta} f\big(x_{n,j}^{\alpha,\beta}\big)\,, \quad \lambda_{n,j}^{\alpha,\beta} = \frac{1}{\pi} \int_{-1}^{1} l_{n,j}^{\alpha,\beta}\, \sigma_{\alpha,\beta}\, dx$$

is called Gaussian quadrature rule for $\pi^{-1} \int_{-1}^{1} \cdot\, \sigma_{\alpha,\beta}\, dx$ or Gaussian quadrature rule with respect to $\sigma_{\alpha,\beta}$. The numbers $x_{n,j}^{\alpha,\beta}$ and $\lambda_{n,j}^{\alpha,\beta}$ are called knots and weights, respectively, of the quadrature rule $Q_n^{\alpha,\beta}$.

The proof of the above stated fact that all zeros of $p_n^{\alpha,\beta}$ are simple and lie in $(-1, 1)$ can be found, for example, in [F]. Further, it is well known that $Q_n^{\alpha,\beta}$ is exact on Π_{2n}, i.e.,

$$Q_n^{\alpha,\beta} p = \frac{1}{\pi} \int_{-1}^{1} p(x)\, \sigma_{\alpha,\beta}(x)\, dx \quad \text{for all} \quad p \in \Pi_{2n}\,. \tag{5.18}$$

Particularly, $\lambda_{n,k}^{\alpha,\beta} = Q_n^{\alpha,\beta} \big(l_{n,k}^{\alpha,\beta}\big)^2 = \pi^{-1} \int_{-1}^{1} \big(l_{n,k}^{\alpha,\beta}\big)^2(x)\, \sigma_{\alpha,\beta}(x)\, dx > 0$ for all $k = 1, \ldots, n$.

Now we suppose that we have given weights u and w such that the assumptions (5.2)–(5.8) and the solvability condition (5.16) are satisfied. We want to apply Theorems 4.6 and 4.10 to an approximation method of the type (4.11) for equation (5.14), where we take $\mathbf{X} = \mathbf{C}_{u(\alpha,\beta)}$ and $\mathbf{Y} = \mathbf{C}_u$. Hence, we have to define appropriate projections $P_n \in \mathcal{L}\big(\mathbf{C}_{u(\alpha,\beta)}\big)$ and $L_n \in \mathcal{L}\big(\mathbf{C}_u\big)$ with not too big operator norms. At the beginning of this section we have already mentioned that, from the computational point of view, it would be advantageous to take $L_n = L_{n(\alpha,\beta)}^{-\alpha,-\beta}$ and to define P_n in such a way that only the values $f\big(x_{n,j}^{\alpha,\beta}\big)$ are necessary for the determination of $P_n f$. Moreover, we will see that, for the construction of good (in the sense of assumptions (i) and (ii) of Theorem 4.6) approximations $\mathcal{K}_n$ and $\mathcal{H}_n$ of $\mathcal{K}$ and $\mathcal{H}$, respectively, the image space $\mathbf{X}_n$ of P_n should consist of polynomials (since we want to use Gaussian quadrature rules to approximate the integrals $(\mathcal{K} f_n)(x)$, $f_n \in \mathbf{X}_n$, and $(\mathcal{H} f_n)(x)$, $f_n \in \mathbf{X}_n$, respectively). This suggests to take $P_n = L_n^{\alpha,\beta}$. But, in general, there appear two problems with these choices of L_n and P_n. Let us first consider the case of a weight u without zeros inside $(-1, 1)$ (i.e., $N = 0$). Although $\mathbf{C}_{u(\alpha,\beta)}$ consists of continuous functions on $(-1, 1)$ in this case, which implies that $L_n^{\alpha,\beta}$ is well-defined on $\mathbf{C}_{u(\alpha,\beta)}$, it might happen that the corresponding operator norm of $L_n^{\alpha,\beta}$ converges too fast to infinity (i.e. $\|L_n^{\alpha,\beta}\| \ge c\, n^{\mu}$ with some $\mu > 0$ which is so big that the assumption (iii) of Theorem 4.6 cannot be satisfied; see [MR] and [S]). The same problem may appear for the projections $L_{n(\alpha,\beta)}^{-\alpha,-\beta}$ in the space $\mathbf{C}_u$. For this reason, we introduce the following modified interpolation operators (see [MR] or [CJLM]) which depend on two parameters $r, s \in \{0, 1\}$ which can be chosen in such a way that this problem disappears (as we will see later).

Definition 5.10 *Let $r, s \in \{0, 1\}$, $\alpha, \beta \in (-1, 1) \setminus \{0\}$, $n \ge 2$, and*

$$x_{n,0}^{\alpha,\beta} := \frac{\min\big\{x_{n,1}^{\alpha,\beta},\, x_{n(\alpha,\beta),1}^{-\alpha,-\beta}\big\} - 1}{2}\,, \quad x_{n,n+1}^{\alpha,\beta} := \frac{\max\big\{x_{n,n}^{\alpha,\beta},\, x_{n(\alpha,\beta),n(\alpha,\beta)}^{-\alpha,-\beta}\big\} + 1}{2}\,. \tag{5.19}$$

By $L_{n,r,s}^{\alpha,\beta}$ *we denote the Lagrangian interpolation operator with respect to* $\{x_{n,j}^{\alpha,\beta}\}_{j=1-s}^{n+r}$, *i.e.,* $L_{n,r,s}^{\alpha,\beta} f$ *is that polynomial of degree* $< n+r+s$ *which satisfies* $\big(L_{n,r,s}^{\alpha,\beta} f\big)\big(x_{n,j}^{\alpha,\beta}\big) = f\big(x_{n,j}^{\alpha,\beta}\big)$, $j = 1-s, \ldots, n+r$.

Now we consider the case of a weight u with zeros x_i inside $(-1,1)$. Since the elements f of $\mathbf{C}_{u(\alpha,\beta)}$ and $\mathbf{C}_u$, respectively, may have singularities in these x_i, we get problems with the computation of $L_{n,r,s}^{\alpha,\beta} f$ and $L_{n,r,s}^{-\alpha,-\beta} f$, respectively, if there is some $x_{n,j}^{\alpha,\beta}$ ($x_{n,j}^{-\alpha,-\beta}$) which is very close (or even equal) to one of the points x_i ($i \in \{1,\ldots,N\}$). To solve this problem, we have to define a second modification of the projections $L_{n(\alpha,\beta)}^{-\alpha,-\beta}$ and $L_n^{\alpha,\beta}$, respectively. For this aim, we remark that, for every $n \ge n_0$, the mappings

$$\begin{aligned} & i \in \{1,\ldots,N\} \;\to\; m(i) \in \{1,\ldots,n\} : \; \big|x_i - x_{n,m(i)}^{\alpha,\beta}\big| = \min_{j=1,\ldots,n} \big|x_i - x_{n,j}^{\alpha,\beta}\big| \\ & \qquad\qquad \text{(if there are two possible indices } m(i)\text{, take the smaller one)}\,, \\ & i \in \{1,\ldots,N\} \;\to\; k(i) \in \{1,\ldots,n(\alpha,\beta)\} : \; \big|x_i - x_{n(\alpha,\beta),k(i)}^{-\alpha,-\beta}\big| = \min_{j=1,\ldots,n(\alpha,\beta)} \big|x_i - x_{n(\alpha,\beta),j}^{-\alpha,-\beta}\big| \\ & \qquad\qquad \text{(if there are two possible indices } k(i)\text{, take the smaller one)} \end{aligned} \tag{5.20}$$

are injective, since, for every $i \in \{0,\ldots,N\}$,

$$[x_i, x_{i+1}] \text{ contains at least two zeros of } p_n^{\alpha,\beta} \text{ and } p_{n(\alpha,\beta)}^{-\alpha,-\beta}, \text{ respectively,} \tag{5.21}$$

because of

$$x_{n,j+1}^{\alpha,\beta} - x_{n,j}^{\alpha,\beta} \le c\, n^{-1}\,, \;\; j = 1,\ldots,n-1\,, \quad 1 - x_{n,n}^{\alpha,\beta}\,, 1 + x_{n,1}^{\alpha,\beta} \sim n^{-2} \tag{5.22}$$

(see [N, Theorem 9.22]). Thus, if we modify the operator $L_{n,r,s}^{\alpha,\beta}$ $\big(L_{n(\alpha,\beta),r,s}^{-\alpha,-\beta}\big)$ simply by omitting the interpolation knots $x_{n,m(i)}^{\alpha,\beta}$ $\big(x_{n(\alpha,\beta),k(i)}^{-\alpha,-\beta}\big)$, $i = 1,\ldots,N$, then the dimension of its image space is reduced by N.

It turns out that, finally, the following operators are well appropriate for our purposes.

Definition 5.11 *Take* u *as in (5.2) and* α, β, $x_{n,0}^{\alpha,\beta}$, $x_{n,n+1}^{\alpha,\beta}$ *as in Definition 5.10. Set*

$$r = r(\alpha,\beta,u) := \left\{ \begin{array}{ll} 0\,, & \tau_{N+1} \ge \dfrac{\alpha}{2} + \dfrac{1}{4} \\[2ex] 1\,, & \tau_{N+1} < \dfrac{\alpha}{2} + \dfrac{1}{4} \end{array} \right\}, \quad s = s(\alpha,\beta,u) := \left\{ \begin{array}{ll} 0\,, & \tau_0 \ge \dfrac{\beta}{2} + \dfrac{1}{4} \\[2ex] 1\,, & \tau_0 < \dfrac{\beta}{2} + \dfrac{1}{4} \end{array} \right\}.$$

By $L_{n,u}^{\alpha,\beta}$ *(n large, such that (5.21) holds for all i) we denote the Lagrangian interpolation operator with respect to the knots* $\{x_{n,j}^{\alpha,\beta}\}_{j=1-s}^{n+r} \setminus \{x_{n,m(i)}^{\alpha,\beta}\}_{i=1}^{N}$, *i.e.,* $L_{n,u}^{\alpha,\beta} f$ *is defined by*

$$L_{n,u}^{\alpha,\beta} f \in \Pi_{n+r+s-N} \;\; \textit{and} \;\; \big(L_{n,u}^{\alpha,\beta} f\big)\big(x_{n,j}^{\alpha,\beta}\big) = f\big(x_{n,j}^{\alpha,\beta}\big)\,, \;\; j \in \{1-s,\ldots,n+r\} \setminus \{m(i)\}_{i=1}^{N}\,.$$

For $n \ge n_0$ *(*n_0 *such that (5.21) is satisfied for* $n \ge n_0$ *and* $i = 0,\ldots,N$*) we define*

$$P_n = L_{n,\,u(\alpha,\beta)}^{\alpha,\beta} \quad \textit{and} \quad L_n = L_{n(\alpha,\beta),\,u}^{-\alpha,-\beta}\,.$$

For $n < n_0$ *we set* $P_n = L_n = 0$.

We remark that $\widetilde{k}(-\alpha) = 1 - \widetilde{k}(\alpha)$ and $\widetilde{l}(-\beta) = 1 - \widetilde{l}(\beta)$ imply $m(-\alpha,-\beta) = n$ for $m = n(\alpha,\beta)$ and, consequently,

$$x^{-\alpha,-\beta}_{n(\alpha,\beta),0} = x^{\alpha,\beta}_{n,0}\,, \quad x^{-\alpha,-\beta}_{n(\alpha,\beta),n(\alpha,\beta)+1} = x^{\alpha,\beta}_{n,n+1} \qquad \text{for all } \; n \geq 2\,.$$

Further, the numbers $r(\alpha,\beta,u(\alpha,\beta))$ and $r(-\alpha,-\beta,u)$ $\big(s(\alpha,\beta,u(\alpha,\beta))$ and $s(-\alpha,-\beta,u)\big)$ coincide. Thus, P_n and L_n $(n \geq n_0)$ are the Lagrangian interpolation operators with respect to

$$\big\{x^{\alpha,\beta}_{n,j}\big\}^{n+r}_{j=1-s} \setminus \big\{x^{\alpha,\beta}_{n,m(i)}\big\}^{N}_{i=1} \quad \text{and} \quad \big\{x^{-\alpha,-\beta}_{n(\alpha,\beta),j}\big\}^{n(\alpha,\beta)+r}_{j=1-s} \setminus \big\{x^{-\alpha,-\beta}_{n(\alpha,\beta),k(i)}\big\}^{N}_{i=1}\,,$$

respectively, where

$$r = \left\{ \begin{array}{ll} 0\,, & \tau_{N+1} \geq -\dfrac{\alpha}{2} + \dfrac{1}{4} \\ 1\,, & \tau_{N+1} < -\dfrac{\alpha}{2} + \dfrac{1}{4} \end{array} \right\} \quad \text{and} \quad s = \left\{ \begin{array}{ll} 0\,, & \tau_0 \geq -\dfrac{\beta}{2} + \dfrac{1}{4} \\ 1\,, & \tau_0 < -\dfrac{\beta}{2} + \dfrac{1}{4} \end{array} \right\}. \tag{5.23}$$

Clearly, the operators P_n and L_n are projections from $\mathbf{X} = \mathbf{C}_{u(\alpha,\beta)}$ and $\mathbf{Y} = \mathbf{C}_u$ onto

$$\mathbf{X}_n = \Pi_{n+r+s-N} \quad \text{and} \quad \mathbf{Y}_n = \Pi_{n(\alpha,\beta)+r+s-N} \qquad (n \geq n_0)\,, \tag{5.24}$$

respectively. (For $n < n_0$ we have $\mathbf{X}_n = \mathbf{Y}_n = \{0\}$.) Hence, in view of (4.11), we look for approximate solutions

$$f_n \in \Pi_{n+r+s-N} \; (n \geq n_0) \;\text{ such that }\; \big[I + P_n(\mathcal{H}_n + \widehat{\mathcal{A}}\,L_n\mathcal{K}_n)\big]\, f_n = P_n\widehat{\mathcal{A}}\,L_n g \tag{5.25}$$

in order to solve (5.14) numerically. Here $\mathcal{H}_n$ and $\mathcal{K}_n$ are appropriate approximations of $\mathcal{H}$ and $\mathcal{K}$, respectively. The quadrature method, which we will study in all of what follows, corresponds to that operators $\mathcal{K}_n$ and $\mathcal{H}_n$ which are obtained by replacing the integral $\pi^{-1}\int_{-1}^{1} \,.\, \sigma_{\alpha,\beta}(t)\,dt$ by the quadrature rule $Q_n^{\alpha,\beta}(\,.\,)$ (in the case of $\mathcal{K}_n$) and a modification of $Q_n^{\alpha,\beta}(\,.\,)$ (in the case of $\mathcal{H}_n$). More precisely,

$$(\mathcal{K}_n f)(x) = \sum_{j=1}^{n} \lambda^{\alpha,\beta}_{n,j}\, k\big(x, x^{\alpha,\beta}_{n,j}\big)\, f\big(x^{\alpha,\beta}_{n,j}\big)\,, \tag{5.26}$$

$$(\mathcal{H}_n f)(x) = \sum_{\substack{j=1 \\ j\neq d(x)}}^{n} \lambda^{\alpha,\beta}_{n,j}\, h\big(x, x^{\alpha,\beta}_{n,j}\big)\, f\big(x^{\alpha,\beta}_{n,j}\big) + \widetilde{b}(x) \sum_{j=1}^{n} \lambda^{\alpha,\beta}_{n,j}\, b\big(x^{\alpha,\beta}_{n,j}\big)\, f\big(x^{\alpha,\beta}_{n,j}\big)\,, \tag{5.27}$$

where $\widetilde{b}(x)$ and $h(x,t)$ are defined as follows,

$$\begin{aligned} \widetilde{b}(x) &= -\frac{\pi\,\widetilde{k}\,\widetilde{l}\; b(x)}{r^2(x)\cos^2\big(\ell_{\alpha,\beta}(x)\big)\int_{-1}^{1}\sigma_{\alpha,\beta}(\tau)\,d\tau}\,, \\ h(x,t) &= \frac{1}{r^2(x)\cos^2\big(\ell_{\alpha,\beta}(x)\big)}\left[\frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\,b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\,b(x)}{t-x}\right], \end{aligned}$$

and $d(x)$ depends on the smoothness of the coefficient functions a, b of the operator A: We have $a, b \in \mathbf{H}^s([-1,1])$ and the value of s is used in the following definition,

$$\begin{cases} d(x) = 0\,, & s>1\,,\ b(-1)b(1)\neq 0, \\ d(x) = \text{index of that } x_{n,j}^{\alpha,\beta},\ j\in\{0,\dots,N{+}1\}, \text{ which is closest} & \\ \qquad \text{to } x \ \big(\text{choose the smaller } d \text{ if there are two such } x_{n,d}^{\alpha,\beta}\big)\,, & s\le 1 \text{ or } b(-1)b(1)=0. \end{cases}$$

We remark that, in the case $s > 1$, $h(x,x) = \lim_{t\to x} h(x,t)$ is well-defined for $x \in (-1,1)$,

$$h(x,x) = \frac{1}{r^2(x)\cos^2\big(\ell_{\alpha,\beta}(x)\big)}\left[b'(x)\,(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x) - b(x)\,(\widetilde{a}\,\sigma_{-\alpha,-\beta})'(x)\right]. \tag{5.28}$$

We further mention that the possible singularities of $k(x,t)$ on the diagonal $x = t$ cause no trouble for the definition of the operator on the left hand side of equation (5.25), since $k\big(x, x_{n,j}^{\alpha,\beta}\big)$ is well-defined in all interpolation knots of the operator L_n. This is a consequence of the following result:

Proposition 5.12 ([Be], Theorem 2.7) *Let $n \ge 2$, $i \in \{1,\dots,n(\alpha,\beta)\}$, and denote by $j(i)$ the index of that $x_{n,j}^{\alpha,\beta}$, $j \in \{1,\dots,n\}$, which is closest to $x_{n(\alpha,\beta),i}^{-\alpha,-\beta}$. Then,*

$$\left|x_{n(\alpha,\beta),i}^{-\alpha,-\beta} - x_{n,j(i)}^{\alpha,\beta}\right| \sim v^{-\alpha,-\beta}\big(x_{n,j(i)}^{\alpha,\beta}\big)\,\lambda_{n,j(i)}^{\alpha,\beta}\,,$$

where the constants in this relation do not depend on n and i.

Later we will show how the method (5.25) can be numerically realized. But first we investigate the convergence.

5.3 Convergence of the quadrature method

Take the assumptions and notation of Proposition 5.6 and let P_n, L_n, and $\mathcal{K}_n$, $\mathcal{H}_n$ be given in Definition 5.11 and (5.26),(5.27), respectively.

We will apply Theorems 4.6 and 4.10 to prove the stability and convergence of the quadrature method (5.25). For sequences

$$b_n = 2(1-2^{-n})\max_{1\le m\le n} m^{\gamma_1}\log_2^{\delta_1}(m+1)\,,\quad c_n = 2(1-2^{-n})\max_{1\le m\le n} m^{\gamma_2}\log_2^{\delta_2}(m+1)$$

($\gamma_i > 0$, $\delta_i \in \mathbb{R}$; see Remark 1.10) we use the notation $[(\gamma_1,\delta_1),(\gamma_2,\delta_2)]$–regularizer instead of $(\mathcal{B},\mathcal{C})$–regularizer. First we show that, for given $\gamma > 0$, $\delta \in \mathbb{R}$, there exist certain $\widetilde{\gamma} > 0$ and $\widetilde{\delta} \in \mathbb{R}$ (the precise definition of these numbers is given below) such that

$$\widehat{\mathcal{A}} \text{ is a left } \big[(\gamma,\delta),(\widetilde{\gamma},\widetilde{\delta})\big]\text{–regularizer of } \mathcal{A} \in \mathcal{L}\big(\mathbf{C}^0_{u(\alpha,\beta)}, \mathbf{C}_u\big) \cap \mathcal{L}\big(\mathbf{C}^*_{u(\alpha,\beta)}, \mathbf{C}^0_u\big)\,, \tag{5.29}$$

where $\mathbf{C}^*_{u(\alpha,\beta)} = (\mathbf{C}_{u(\alpha,\beta)})_1^{\mathcal{D}}(\{\Pi_n\})$ with $\mathcal{D} = \{\log_2^2(n+1)\}$. We remark that it is justified to consider the approximation spaces $\mathbf{C}^0_{\dots}$, $\mathbf{C}^*_{\dots}$, $\mathbf{C}^{\gamma,\delta}_{\dots}$ defined with the help of errors of best approximation by elements of Π_n instead of $\mathbf{X}_n$ and $\mathbf{Y}_n$ (see (5.24)), respectively, since these spaces are equal (in the sense of equivalent norms) to the corresponding approximation spaces based on $(\mathbf{C}_{u(\alpha,\beta)}, \{\mathbf{X}_n\})$ and $(\mathbf{C}_u, \{\mathbf{Y}_n\})$, respectively. The proof is left to the reader.

The property $\mathcal{A} \in \mathcal{L}\big(\mathbf{C}^0_{u(\alpha,\beta)}, \mathbf{C}_u\big) \cap \mathcal{L}\big(\mathbf{C}^*_{u(\alpha,\beta)}, \mathbf{C}^0_u\big)$ follows from the corresponding property of the operator $A_{\alpha,\beta}$ (Theorem 2.1, (2.43), and the density of Π in $\mathbf{C}^0_{u(\alpha,\beta)}$ yield $A_{\alpha,\beta} \in \mathcal{L}\big(\mathbf{C}^0_{u(\alpha,\beta)}, \mathbf{C}_u\big)$; (2.43) and Theorem 1.21,(ii), applied to $a_n = b_n = \log_2(n+1)$ yield $A_{\alpha,\beta} \in \mathcal{L}\big(\mathbf{C}^*_{u(\alpha,\beta)}, \mathbf{C}^0_u\big)\big)$ and the decomposition

$$\mathcal{A} = \widetilde{a}\,\sigma_{\alpha,\beta}\, I + A_{\alpha,\beta}\, bI\,. \tag{5.30}$$

(Remark that $bI \in \mathcal{L}\big(\mathbf{C}^0_{u(\alpha,\beta)}\big) \cap \mathcal{L}\big(\mathbf{C}^*_{u(\alpha,\beta)}\big)$ because of (1.13) and $\widetilde{a}\,\sigma_{\alpha,\beta}\, I \in \mathcal{L}\big(\mathbf{C}^0_{u(\alpha,\beta)}, \mathbf{C}^0_u\big)$ $\subseteq \mathcal{L}\big(\mathbf{C}^*_{u(\alpha,\beta)}, \mathbf{C}^0_u\big)$ because of Theorem 1.13, Proposition 1.12, and (2.63).) The similar decomposition of $\widehat{\mathcal{A}}$,

$$\widehat{\mathcal{A}} = \frac{\widetilde{a}}{r^2 \sigma_{\alpha,\beta}} I - \frac{b}{r^2 \cos^2(\ell_{\alpha,\beta})} A_{-\alpha,-\beta}\,, \tag{5.31}$$

shows (again by Theorem 2.1, (2.43), and the density of Π in $\mathbf{C}^0_u$) that

$$\widehat{\mathcal{A}} \in \mathcal{L}\big(\mathbf{C}^0_u, \mathbf{C}_{u(\alpha,\beta)}\big)\,.$$

Before we prove the second mapping property of $\widehat{\mathcal{A}}$ which is claimed in Definition 4.2,(ii), we first check under which conditions the operator $\mathcal{H}$ has a mapping property as in Definition 4.2,(iii). Of course, we hope that $\widetilde{\gamma} = \gamma$ appears in (5.29) if a and b are smooth enough. But if $b(-1)b(1) = 0$, then at most $\widetilde{\gamma} = \gamma/2$ can be expected, as we will see later. Now, in view of (2.64) (with $\gamma/2$ instead of γ if $b(-1)b(1) = 0$), it turns out that the number γ which is considered in (5.29) should be chosen small enough such that the following assumptions are satisfied,

$$0 < \gamma < 2\,, \quad \gamma \neq 1\,, \quad \gamma \le 4\tau_{N+1} \text{ if } b(1) = 0\,, \quad \gamma \le 4\tau_0 \text{ if } b(-1) = 0\,, \tag{5.32}$$

$$\begin{cases} a^{([\gamma])}, b^{([\gamma])} \in \mathbf{C}^{\gamma-[\gamma],\min\{\delta+1,0\}} & \text{if} \quad b(-1)b(1) \neq 0\,, \\ a \in \mathbf{C}^{\gamma/2,\min\{\delta+1,0\}},\ b \in \mathbf{C}^{\gamma,\min\{\delta+1,0\}} & \text{if} \quad b(-1)b(1) = 0\,, \end{cases} \tag{5.33}$$

and that, in the case $b(-1)b(1) = 0$, only the values

$$-\tau_{N+1} < \alpha < 1 - 2\tau_{N+1} \text{ if } b(1) = 0\,, \quad -\tau_0 < \beta < 1 - 2\tau_0 \text{ if } b(-1) = 0 \tag{5.34}$$

of α and β, respectively, should be allowed. Then we obtain, by Theorem 2.21 and assertion (iv) of Corollary 1.18 (applied to $f = r^2 \cos^2(\ell_{\alpha,\beta})$ which belongs to $\mathbf{C}^{\gamma,\min\{\delta+1,0\}}$ if $b(-1)b(1) \neq 0$ and to $\mathbf{C}^{\gamma/2,\min\{\delta+1,0\}}$ if $b(-1)b(1) = 0$; see Prop.1.12 and Cor.1.18,(i)) that

$$\mathcal{H} = [r\cos(\ell_{\alpha,\beta})]^{-2}\, \sigma_{-\alpha,-\beta}\, r^2 H \sigma_{\alpha,\beta}\, I \in \mathcal{L}\big(\mathbf{C}_{u(\alpha,\beta)}, \mathbf{C}^{\widetilde{\gamma},\widetilde{\delta}}_{u(\alpha,\beta)}\big)\,, \tag{5.35}$$

where

$$\widetilde{\gamma} = \begin{cases} \gamma, & \text{if} \quad b(1) \neq 0 \text{ and } b(-1) \neq 0, \\ \gamma/2, & \text{if} \quad b(1) = 0 \text{ or } b(-1) = 0, \end{cases} \tag{5.36}$$

$$\widetilde{\delta} = \begin{cases} \delta, & \text{if (5.32),(5.33) are satisfied with } \gamma \text{ replaced by some } \gamma' > \gamma, \\ \min\{\delta, -1\}, & \text{otherwise.} \end{cases} \tag{5.37}$$

Also the second condition of Definition 4.2, (iii), $\widehat{\mathcal{A}}\mathcal{A}f = f + \mathcal{H}f$ for all $f \in \Pi$, is satisfied in view of Theorem 2.21. It remains to prove

$$\widehat{\mathcal{A}} \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta+1}, \mathbf{C}_{u(\alpha,\beta)}^{\widetilde{\gamma},\widetilde{\delta}}\big)\,. \tag{5.38}$$

From (2.44), Proposition 1.12, and assertions (i) and (iv) of Corollary 1.18 it follows

$$\frac{b}{r^2\cos^2(\ell_{\alpha,\beta})} A_{-\alpha,-\beta} \in \mathcal{L}\big(\mathbf{C}_u^{\gamma,\delta+1}, \mathbf{C}_{u(\alpha,\beta)}^{\widetilde{\gamma},\widetilde{\delta}}\big)\,.$$

That the same is true for the first addend on the right hand side of (5.31) is a consequence of Lemma 2.28, of the following lemma, and of the continuous embedding

$$\begin{cases} \mathbf{C}_u^{\gamma,\delta+1} \subseteq \mathbf{C}_u^{\gamma,\widetilde{\delta}+1} & \text{if} \quad b(1) \neq 0 \;\text{ and }\; b(-1) \neq 0\,, \\ \mathbf{C}_u^{\gamma,\delta+1} \subseteq \mathbf{C}_{u(-\gamma/4,0)}^{\gamma/2,\delta+1} \subseteq \mathbf{C}_{u(-\gamma/4,0)}^{\gamma/2,\widetilde{\delta}+1} & \text{if} \quad b(1) = 0 \;\text{ and }\; b(-1) \neq 0\,, \\ \mathbf{C}_u^{\gamma,\delta+1} \subseteq \mathbf{C}_{u(0,-\gamma/4)}^{\gamma/2,\delta+1} \subseteq \mathbf{C}_{u(0,-\gamma/4)}^{\gamma/2,\widetilde{\delta}+1} & \text{if} \quad b(1) \neq 0 \;\text{ and }\; b(-1) = 0\,, \\ \mathbf{C}_u^{\gamma,\delta+1} \subseteq \mathbf{C}_{u(-\gamma/4,-\gamma/4)}^{\gamma/2,\delta+1} \subseteq \mathbf{C}_{u(-\gamma/4,-\gamma/4)}^{\gamma/2,\widetilde{\delta}+1} & \text{if} \quad b(1) = 0 \;\text{ and }\; b(-1) = 0 \end{cases}$$

(see assertion (vii) of Theorem 1.11).

Lemma 5.13 *If (5.32) and (5.33) are satisfied, then*

$$\frac{\widetilde{a}}{r^2\sigma_{\alpha,\beta}} \in \begin{cases} \mathbf{C}_{\alpha,\beta}^{\gamma,\min\{\delta+1,\,0\}} & \textit{if} \quad b(1) \neq 0 \;\textit{ and }\; b(-1) \neq 0\,, \\ \mathbf{C}_{\alpha+(\gamma/4),\beta}^{\gamma/2,\min\{\delta+1,\,0\}} & \textit{if} \quad b(1) = 0 \;\textit{ and }\; b(-1) \neq 0\,, \\ \mathbf{C}_{\alpha,\beta+(\gamma/4)}^{\gamma/2,\min\{\delta+1,\,0\}} & \textit{if} \quad b(1) \neq 0 \;\textit{ and }\; b(-1) = 0\,, \\ \mathbf{C}_{\alpha+(\gamma/4),\beta+(\gamma/4)}^{\gamma/2,\min\{\delta+1,\,0\}} & \textit{if} \quad b(1) = 0 \;\textit{ and }\; b(-1) = 0\,, \end{cases}$$

where $\mathbf{C}_{\rho,\tau}^{\gamma,\delta} := \mathbf{C}_{\psi}^{\gamma,\delta}$ *with* $\psi = v^{\rho,\tau}$ *(see Remark 5.2).*

Let us summarize: If $\widehat{\mathcal{A}}$ is defined as in Section 5.1, then the conditions (5.32), (5.33), and (5.34) ensure that (5.29) is satisfied with $\widetilde{\gamma}$ and $\widetilde{\delta}$ from (5.36) and (5.37), respectively. Moreover, we know from Theorems 3.1 and 3.5 that the assumptions (5.4) and (5.5) on $k(x,t) = k_1(x,t) + k_2(x,t)$ together with the conditions

$$\operatorname*{ess\,sup}_{t\in[-1,1]} \big\|u^{-1}(\,.\,)\,h_1(\,.\,,t)\big\|_{u,\gamma,\delta+1} < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in[-1,1]} \big\|h_2(\,.\,,t)\big\|_{\gamma,\delta+2} < \infty \tag{5.39}$$

imply that $\mathcal{K} = K\sigma_{\alpha,\beta}\,I$ belongs to $\mathcal{L}\big(\mathbf{C}_{u(\alpha,\beta)}, \mathbf{C}_u^{\gamma,\delta+1}\big)$.

To show that Theorem 4.10 is applicable to our approximation method (5.25) it remains to check that the assumptions (i)–(iii) of Theorem 4.6 are satisfied. We will do this later. Here we just state the final result.

Theorem 5.14 *Let the conditions (5.2)–(5.5) on $u(x)$ and $k(x,t) = k_1(x,t) + k_2(x,t)$ as well as the conditions (5.32) and (5.33) on γ, δ, a, and b be satisfied. Further, suppose that (5.34) and (5.39) hold true and that*

$$\operatorname*{ess\,sup}_{x\in[-1,1]} \left\| h_1(x,\,.\,)\, w^{-1}(\,.\,) \right\|_{w,\gamma,\delta+1} < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{x\in[-1,1]} \left\| h_2(x,\,.\,) \right\|_{\gamma,\delta+2} < \infty\,. \tag{5.40}$$

Define $\widetilde{\gamma}$ and $\widetilde{\delta}$ by (5.36) and (5.37), respectively. If the exponents τ_i belonging to the inner zeros $x_i \in (-1,1)$ of u are chosen close enough to 1 such that

$$2 \max_{i=1,\dots,N} (1-\tau_i) < \gamma\,,$$

if $g \in \mathbf{C}_u^{\gamma,\delta+1}$, and if $(I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K})f = 0$ possesses only the trivial solution in $\mathbf{H}_{u(\alpha,\beta)}$, then there exist uniquely determined solutions $f^* \in \mathbf{C}_{u(\alpha,\beta)}$ and $f_n^* \in \Pi_{n+r+s-N}$ ($n \ge n_0$ large enough) of (5.14) and (5.25), respectively, where

$$\|f^* - f_n^*\|_{u(\alpha,\beta)} \le c \left(\frac{\ln^{1-\widetilde{\delta}} n}{n^{\widetilde{\gamma} - \max_{i=1,\dots,N}(1-\tau_i)}} + \frac{\ln^{2-\delta} n}{n^{\gamma - 2\max_{i=1,\dots,N}(1-\tau_i)}} \right) \|g\|_{u,\gamma,\delta+1}\,, \quad n \ge n_0\,,$$

with constants n_0 and c which are independent of n and g.

Remark 5.15 *Let $b(-1)b(1) \ne 0$. If $\gamma > 1$, then $a, b \in \mathbf{H}^s([-1,1])$ for some $s > 1$ (see assertion (i) of Proposition 1.19), i.e., in this case we set $d(x) = 0$ in (5.27). If $\gamma < 1$, then it may also happen that a' and b' are Hölder continuous. In such a case both possible definitions of $d(x)$ are allowed.*

Remark 5.16 *Let the assumptions of Theorem 5.14 be satisfied. If the initial equation (5.1) possesses a solution ϕ in $\mathbf{H}_u$, then this solution is uniquely determined by $\phi = \sigma_{\alpha,\beta} f^*$ (see Proposition 5.6) and approximate solutions are given by $\phi_n = \sigma_{\alpha,\beta} f_n^*$, where the above estimate for $\|f^* - f_n^*\|_{u(\alpha,\beta)}$ yields a corressponding estimate for $\|\phi - \phi_n\|_u$.*

Remark 5.17 *Let us consider an operator A with constant coefficients,*

$$A = \pm r \left[\sin\left(\pi\alpha_0 + \frac{\pi}{2}\right) I + \cos\left(\pi\alpha_0 + \frac{\pi}{2}\right) S \right],$$

where $r = \mathrm{const} > 0$ and $\alpha_0 \in (-1,0)$. Then $\beta_0 = -\alpha_0 - 1$, $\ell_{\alpha,\beta} = \pi\alpha_0 + \frac{\pi}{2}$, $\varkappa_0 = 1$, and $\widetilde{a} = 0$. From (2.36), (5.30), and (5.31) it follows that $\sigma_{\alpha,\beta}$, $\sigma_{-\alpha,-\beta}$, $\mathcal{A}$, and $\widehat{\mathcal{A}}$ are, up to constant factors, equal to $v^{\alpha,\beta}$, $v^{-\alpha,-\beta}$, $A_{\alpha,\beta}$, and $A_{-\alpha,-\beta}$, respectively. Let us suppose that $\tau_0 \ge -\beta_0$ or $\tau_{N+1} \ge -\alpha_0$ and that the initial equation $(\mathcal{A} + \mathcal{K})f = g$ has exactly one solution $f \in \mathbf{H}_{u(\alpha,\beta)}$ which satisfies

$$\int_{-1}^{1} f(x)\, v^{\alpha,\beta}(x)\, dx = 0 \quad \text{if} \quad \tau_0 \ge -\beta_0 \ \text{ and } \ \tau_{N+1} \ge -\alpha_0\,. \tag{5.41}$$

(Remark that $\ker \mathcal{A} = \Pi_1$ if $\tau_0 \ge -\beta_0$ and $\tau_{N+1} \ge -\alpha_0$; see Proposition 2.17.) Then $\mathcal{H}f = 0$ for this solution f and this suggest to consider the equations (5.14) and (5.25) without $\mathcal{H}$ and $\mathcal{H}_n$, respectively, i.e.,

$$f \in \mathbf{C}_{u(\alpha,\beta)} : \ (I + \widehat{\mathcal{A}}\mathcal{K})f = \widehat{\mathcal{A}}\, g\,, \qquad f_n \in \Pi_{n+r+s-N} : \ (I + \widehat{\mathcal{A}}\, L_n \mathcal{K}_n) f_n = \widehat{\mathcal{A}}\, L_n g\,, \tag{5.42}$$

where we took (2.43) into account. By Proposition 2.17 and Remark 5.7, (5.42) holds if and only if $f \in \mathbf{H}_{u(\alpha,\beta)}$ *and* $f_n \in \Pi_{n+r+s-N}$ *satisfy (5.41) and* $(\mathcal{A}+\mathcal{K})f = g$, $(\mathcal{A}+L_n\mathcal{K}_n)f_n = L_n g$. *Since* P_n *does not appear in (5.42), one can use (4.24) without the factor* $\|P_n\|$. *This shows, in view of the proof of Theorem 5.14, that the error estimate*

$$\|f^* - f_n^*\|_{u(\alpha,\beta)} \leq c \frac{\ln^{1-\delta} n}{n^{\gamma - \max_{i=1,\ldots,N}(1-\tau_i)}} \|g\|_{u,\gamma,\delta+1}\,, \quad n \geq n_0\,,$$

holds true for the unique solutions of (5.42), where we can even consider arbitrary $\gamma > 0$, *since the restriction* $\gamma \in (0,2) \setminus \{1\}$ *is only needed for the investigation of* $\mathcal{H}$ *an* $\mathcal{H}_n$.

At the end of this section we give a non-trivial corollary of Theorem 5.14 (and its proof) which shows that the assumptions on $k(x,t)$ and g can be weakened and that the second addend in the error estimate can be omitted if powers of $\ln n$ are replaced by n^ε.

Corollary 5.18 *Let the assumptions (5.3)–(5.5) be satisfied with u replaced by*

$$\widetilde{u}(x) = v^{\tau_{N+1},\tau_0}(x) \prod_{j=1}^{N} |x - x_j|\,,$$

where x_j, τ_0, *and* τ_{N+1} *fulfill the assumptions given in (5.2). Further, let* α, β, γ, δ, a, b, *and the part* $k_2(x,t)$ *of* $k(x,t) = k_1(x,t) + k_2(x,t)$ *satisfy the assumptions from (5.32), (5.33), (5.34), (5.39), and (5.40), while* $k_1(x,t)$ *only has to satisfy*

$$\operatorname*{ess\,sup}_{t\in[-1,1]} \left\|k_1(\,.\,,t)\, w(t)\right\|_{\widetilde{u},\gamma,\delta+1} < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{x\in[-1,1]} \left\|\widetilde{u}(x)\, k_1(x,\,.\,)\right\|_{w,\gamma,\delta+1} < \infty\,. \tag{5.43}$$

Then, $I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K}$ *has a trivial kernel in one of the spaces* $\mathbf{C}_{u(\alpha,\beta)}$ *with*

$$u(x) = \prod_{j=0}^{N+1} |x - x_j|^{\tau_j} \quad \text{satisfying} \quad \max\left\{0,\, 1 - \widetilde{\gamma}\right\} < \tau_j < 1\,, \;\; j = 1, \ldots, N \tag{5.44}$$

if and only if it has a trivial kernel in each of these spaces. In this case the assumption $g \in \mathbf{C}_{\widetilde{u}}^{\gamma,\delta+1}$ *implies that (5.14) and (5.25) possess unique solutions* $f^* \in \mathbf{C}_{u(\alpha,\beta)}$ *and* $f_n^* \in \Pi_{n+r+s-N}$ *(*$n \geq n_0$ *large enough), respectively, where, for every fixed* $\varepsilon > 0$,

$$\|f^* - f_n^*\|_{u(\alpha,\beta)} \leq \frac{c}{n^{\widetilde{\gamma} - \max_{i=1,\ldots,N}(1-\tau_i) - \varepsilon}} \|g\|_{\widetilde{u},\gamma,\delta+1}\,, \quad n \geq n_0\,, \tag{5.45}$$

with constants n_0 *and* c *which are independent of* n *and* g.

Remark 5.19 *Clearly, in the assumptions of the above corollary we may replace all spaces* $\mathbf{C}_{\cdot\cdot}^{\gamma,\cdot}$ *and their norms by the spaces* $\mathbf{C}_{\cdot\cdot}^{\gamma,0}$ *and their norms. Moreover, the approximation spaces based on* $\mathbf{C}_{\widetilde{u}}$ *can be characterized with the help of approximation spaces based on* $\mathbf{C}_{\tau_{N+1},\tau_0} := \mathbf{C}_{v^{\tau_{N+1},\tau_0}}$. *Indeed, from the identity* $\|(f - p_n)\widetilde{u}\| = \|(fP - p_nP)v^{\tau_{N+1},\tau_0}\|$, $P(x) = \prod_{i=1}^{N}(x - x_i)$, *it follows easily that* $f \in \mathbf{C}_{\widetilde{u}}$ *belongs to* $\mathbf{C}_{\widetilde{u}}^{\gamma,\delta}$ *if and only if* $fP \in \mathbf{C}_{\tau_{N+1},\tau_0}^{\gamma,\delta}$ *and* $(fP)(x_i) = 0$ *for all* $i = 1, \ldots, N$. *Furthermore, the norms of* f *and* fP *in the corresponding approximation spaces are equivalent for all* $f \in \mathbf{C}_{\widetilde{u}}$ *with* $(fP)(x_i) = 0$, $i = 1, \ldots, N$.

5.4 Computational aspects

5.4.1 The linear system

Let us fix a sufficiently large $n \in \mathbb{N}$. To solve (5.25) numerically we determine an equivalent linear system of $n+r+s-N$ equations for the $n+r+s-N$ unknown function values

$$\phi_k := f_n\big(x_{n,k}^{\alpha,\beta}\big)\,, \quad k \in \{1-s,\dots,n+r\} \setminus \{m(1),\dots,m(N)\}$$

of the solution $f_n \in \Pi_{n+r+s-N}$ of (5.25). (Clearly, f_n is uniquely determined by these values.) We will see that the computation of the coefficient matrix of this system requires the knowledge of the knots and weights of the Gaussian quadrature rules with respect to $\sigma_{\alpha,\beta}$ and $\sigma_{-\alpha,-\beta}$ of order n and

$$m := n(\alpha,\beta)\,,$$

respectively. These knots and weights as well as certain other parameters have to be computed in advance. The realization of this preprocessing will be studied in the second part of this section. Here we give only a list of the parameters which have to be computed:

Preprocessing. Compute

$$\sigma_k := \lambda_{n,k}^{\alpha,\beta}\,,\ k=1,\dots,n\,,\quad \mu_k := \lambda_{m,k}^{-\alpha,-\beta}\,,\ k=1,\dots,m\,,$$

$$t_k := x_{n,k}^{\alpha,\beta}\,,\ k=1-s,\dots,n+r\,,\quad y_k := x_{m,k}^{-\alpha,-\beta}\,,\ k=1-s,\dots,m+r\,,$$

$$k(i)\,,\ i=1,\dots,N\,,\quad m(i)\,,\ i=1,\dots,N\,,$$

$$a_k := \left(\frac{\widetilde{a}}{r^2\sigma_{\alpha,\beta}}\right)(t_k)\,,\ b_k := b(t_k)\,,\ c_k := \widetilde{b}(t_k)\,,\ d_k := -\left(\frac{b}{r^2\cos^2(\ell_{\alpha,\beta})}\right)(t_k)\,,$$
$$k=1-s,\dots,n+r\,,$$

$$h_{j,k} := h(t_j,t_k)\,,\ j\in\{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(N)\},\ k\in\{1,\dots,n\}\setminus\{j\}\,,$$

$$h_{k,k} := \left\{\begin{array}{cc} 0\,, & d(t_k)=k \\ h(t_k,t_k)\,, & d(t_k)=0 \end{array}\right\},\ k\in\{1,\dots,n\}\setminus\{m(1),\dots,m(N)\}\,,$$

$$k_{j,k} := k(y_j,t_k)\,,\ j=\{1-s,\dots,m+r\}\setminus\{k(1),\dots,k(N)\},\ k=1,\dots,n\,,$$

$$p_n^{\alpha,\beta}(t_0),\, p_m^{-\alpha,-\beta}(t_0)\ \text{ if } s=1\,,\quad p_n^{\alpha,\beta}(t_{n+1}),\, p_m^{-\alpha,-\beta}(t_{n+1})\ \text{ if } r=1\,,$$

$$L_{j,k} := l_j^L(t_k)\,,\ j\in\{1-s,\dots,m+r\}\setminus\{k(1),\dots,k(N)\},\ k=1,\dots,n\,,$$

$$L_{j,n+i} := l_j^L(y_{k(i)})\,,\ j\in\{1-s,\dots,m+r\}\setminus\{k(1),\dots,k(N)\},\ i=1,\dots,N\,,$$

$$P_{j,i} := l_j^P(t_{m(i)})\,,\ j\in\{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(N)\},\ i=1,\dots,N\,,$$

where l_j^L and l_j^P denote the fundamental polynomials of Lagrange interpolation with respect to L_n and P_n, respectively, i.e.,

$$l_j^L(x) = \prod_{\substack{l\in\{1-s,\dots,m+r\}\setminus\{k(1),\dots,k(N)\}\\ l\neq j}} \frac{x-y_l}{y_j-y_l}\,,\quad l_j^P(x) = \prod_{\substack{l\in\{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(N)\}\\ l\neq j}} \frac{x-t_l}{t_j-t_l}\,.$$

From the definition of P_n it follows that $f_n \in \Pi_{n+r+s-N}$ satisfies (5.25) if and only if the following equations

$$\Big[f_n + \big(\mathcal{H}_n + \widehat{\mathcal{A}}\, L_n \mathcal{K}_n\big) f_n\Big](t_k) = \big(\widehat{\mathcal{A}}\, L_n g\big)(t_k)\,,\ k \in \{1-s, \ldots, n+r\} \setminus \{m(i)\}_{i=1}^{N} \tag{5.46}$$

hold true. (Remark that $\big[I + P_n\big(\mathcal{H}_n + \widehat{\mathcal{A}}\, L_n\mathcal{K}_n\big)\big] f_n = P_n\big[I + \big(\mathcal{H}_n + \widehat{\mathcal{A}}\, L_n\mathcal{K}_n\big)\big] f_n$.) To determine $(\widehat{\mathcal{A}}\, L_n\mathcal{K}_n f_n)(t_k)$ and $(\widehat{\mathcal{A}}\, L_n g)(t_k)$ we use the following result.

Lemma 5.20 *Let $p \in \Pi_{2m+1}$. Then,*

$$(\widehat{\mathcal{A}}\, p)(t_k) = \left(a_k + d_k \operatorname{sign}(\alpha)\, \frac{p_n^{\alpha,\beta}(t_k)}{p_m^{-\alpha,-\beta}(t_k)}\right) p(t_k) + d_k \sum_{j=1}^{m} \mu_j\, \frac{p(y_j)}{y_j - t_k}$$

for all $k = 1-s, \ldots, n+r$. (Remark that $p_n^{\alpha,\beta}(t_k) = 0$ for $k = 1, \ldots, n$ and that, by Proposition 5.12, $y_j \neq t_k$ for $j = 1, \ldots, m$, $k = 0, \ldots, n+1$.)

If one wants to apply Lemma 5.20 to $p = L_n g$, then one has to compute

$$(L_n g)(t_k) = \begin{cases} g(t_k) & \text{if } \ k \in \{0, n+1\} \cap \{1-s, n+r\}\,, \\ \sum\limits_{j \in \{1-s,\ldots,m+r\} \setminus \{k(1),\ldots,k(N)\}} L_{j,k}\, g(y_j) & \text{if } \ k \in \{1, \ldots, n\}\,, \end{cases}$$

$$(L_n g)(y_j) = \begin{cases} g(y_j) & \text{if } \ j \notin \{k(1), \ldots, k(N)\}\,, \\ \sum\limits_{i \in \{1-s,\ldots,m+r\} \setminus \{k(1),\ldots,k(N)\}} L_{i,n+l}\, g(y_i) & \text{if } \ j = k(l)\,. \end{cases}$$

Set $\mu_0 = \mu_{m+1} = 0$. Then we get $\big(\widehat{\mathcal{A}}\, L_n\, g\big)(t_k) = \sum\limits_{j \in \{1-s,\ldots,m+r\} \setminus \{k(1),\ldots,k(N)\}} a_{k,j}\, g(y_j)$ with

$$a_{k,j} = \left[a_k + \operatorname{sign}(\alpha)\, \frac{d_k\, p_n^{\alpha,\beta}(t_k)}{p_m^{-\alpha,-\beta}(t_k)}\right] \left\{ \begin{array}{ll} L_{j,k}\,, & k \in \{1, \ldots, n\} \\ 0\,, & k = 0,\ j \geq 1 \\ 0\,, & k = n+1,\ j \leq m \\ 1\,, & \text{otherwise} \end{array} \right\} + d_k \left[\frac{\mu_j}{y_j - t_k} + \sum_{i=1}^{N} \frac{\mu_{k(i)}\, L_{j,n+i}}{y_{k(i)} - t_k}\right].$$

Together with (5.26) and (5.27) we conclude that (5.46) can be rewritten as follows.

$$\begin{gathered} \phi_k + \sum_{j=1}^{n} \sigma_j \left(h_{k,j} + c_k\, b_j + \sum_{i \in \{1-s,\ldots,m+r\} \setminus \{k(1),\ldots,k(N)\}} a_{k,i}\, k_{i,j} \right) \phi_j = G_k\,, \\ k \in \{1-s, \ldots, n+r\} \setminus \{m(1), \ldots, m(N)\}\,, \end{gathered} \tag{5.47}$$

where

$$G_k = \sum_{j \in \{1-s,\ldots,m+r\} \setminus \{k(1),\ldots,k(N)\}} a_{k,j}\, g(y_j)\,.$$

If we take into account that, for $i = 1, \ldots, N$,

$$\phi_{m(i)} = (P_n f_n)(t_{m(i)}) = \sum_{j \in \{1-s,\ldots,n+r\} \setminus \{m(1),\ldots,m(N)\}} P_{j,i}\, \phi_j$$

and if we set $\sigma_0 = \sigma_{n+1} = 0$, then we obtain the linear system

$$\phi_k + \sum_{j\in\{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(N)\}} A_{k,j}\,\phi_j = G_k\,, \quad k \in \{1-s,\dots,n+r\}\setminus\{m(i)\}_{i=1}^N \qquad (5.48)$$

for the unknown values ϕ_j, $j \in \{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(N)\}$, where

$$\begin{aligned} A_{k,j} \;&=\; \sigma_j \left(h_{k,j} + c_k\, b_j + \sum_{i\in\{1-s,\dots,m+r\}\setminus\{k(1),\dots,k(N)\}} a_{k,i}\, k_{i,j} \right) \\ &\quad + \sum_{l=1}^N \sigma_{m(l)}\, P_{j,l} \left(h_{k,m(l)} + c_k\, b_{m(l)} + \sum_{i\in\{1-s,\dots,m+r\}\setminus\{k(1),\dots,k(N)\}} a_{k,i}\, k_{i,m(l)} \right). \end{aligned}$$

5.4.2 Preprocessing

It is well known that the zeros of $p_n^{\alpha,\beta}$ are the eigenvalues of the matrix

$$\begin{pmatrix} \alpha_0^{\alpha,\beta} & \beta_1^{\alpha,\beta} & & 0 \\ \beta_1^{\alpha,\beta} & \alpha_1^{\alpha,\beta} & \ddots & \\ & \ddots & \ddots & \beta_{n-1}^{\alpha,\beta} \\ 0 & & \beta_{n-1}^{\alpha,\beta} & \alpha_{n-1}^{\alpha,\beta} \end{pmatrix}, \qquad (5.49)$$

where $\alpha_j^{\alpha,\beta}$ and $\beta_j^{\alpha,\beta}$ are the coefficients of the recurrence formula

$$\beta_0^{\alpha,\beta} p_0^{\alpha,\beta} = 1\,, \quad \beta_{k+1}^{\alpha,\beta} p_{k+1}^{\alpha,\beta}(x) = \big(x - \alpha_k^{\alpha,\beta}\big) p_k^{\alpha,\beta}(x) - \beta_k^{\alpha,\beta} p_{k-1}^{\alpha,\beta}(x) \quad (p_{-1}^{\alpha,\beta} := 0) \qquad (5.50)$$

(see [F, Satz I.2.1] for (5.50) and remark that this formula implies that $\big(p_0^{\alpha,\beta},\dots,p_{n-1}^{\alpha,\beta}\big)(t_k)$, $k = 1,\dots,n$, are the eigenvectors of the matrix (5.49)). Thus, to compute t_k, $k = 1,\dots,n$, and y_k, $k = 1,\dots,m$, one has to determine the recurrence coefficients

$$\alpha_k^{\alpha,\beta},\, \beta_k^{\alpha,\beta}\,, \quad k = 0,\dots,n-1\,, \quad \alpha_k^{-\alpha,-\beta},\, \beta_k^{-\alpha,-\beta}\,, \quad k = 0,\dots,m-1$$

(we also need $\beta_0^{\alpha,\beta}$ and $\beta_0^{-\alpha,-\beta}$ in our later considerations) and the eigenvalues of the corresponding matrices (5.49). There exist efficient and highly precise algorithms for the determination of eigenvalues of symmetric tridiagonal matrices. Here we mention the QR- and QL-algorithm and refer to the literature (e.g., [BMRW]) for the details. Let $P_n^{\alpha,\beta} = \big(1/c_n^{\alpha,\beta}\big)\, p_n^{\alpha\beta}$ (see Definition 5.8). Then, theoretically, the recurrence coefficients $\alpha_k^{\alpha,\beta}$ and $\beta_k^{\alpha,\beta}$ can be determined by the following algorithm:

$$\begin{aligned} &P_{-1}^{\alpha,\beta} \equiv 0,\;\; P_0^{\alpha,\beta} \equiv 1,\;\; a_0 = I_{\alpha,\beta}(1),\;\; b_0 = I_{\alpha,\beta}(x),\;\; \alpha_0^{\alpha,\beta} = \frac{b_0}{a_0},\;\; \beta_0^{\alpha,\beta} = \sqrt{a_0}, \\ &k = 1,2,\dots:\;\; P_k^{\alpha,\beta}(x) = \big(x - \alpha_{k-1}^{\alpha,\beta}\big) P_{k-1}^{\alpha,\beta}(x) - \big(\beta_{k-1}^{\alpha,\beta}\big)^2 P_{k-2}^{\alpha,\beta}(x), \\ &\qquad\qquad a_k = I_{\alpha,\beta}\big((P_k^{\alpha,\beta})^2\big),\;\; b_k^{\alpha,\beta} = I_{\alpha,\beta}\big(x\,(P_k^{\alpha,\beta})^2\big),\;\; \alpha_k^{\alpha,\beta} = \frac{b_k}{a_k},\;\; \beta_k^{\alpha,\beta} = \sqrt{\frac{a_k}{a_{k-1}}}\,, \end{aligned}$$

where $I_{\alpha,\beta}(f) := \pi^{-1}\int_{-1}^{1} f\sigma_{\alpha,\beta}\,dx$. Here we took into account that $a_k = I_{\alpha,\beta}\big(xP_k^{\alpha,\beta}P_{k-1}^{\alpha,\beta}\big)$ (since $I_{\alpha,\beta}\big(x^jP_k^{\alpha,\beta}\big) = 0$ for $j < k$), so that the above formulas for $\alpha_k^{\alpha,\beta}$ and $\beta_k^{\alpha,\beta}$ are obtained from the Fourier expansion of $xP_k^{\alpha,\beta}(x)$ with respect to the orthogonal system $\big\{P_k^{\alpha,\beta}\big\}_{i=0}^{k+1}$. In practice the integrals $I_{\alpha,\beta}(\,.\,)$ are computed approximatively with the help of some quadrature rule

$$I_{\alpha,\beta}(f) \approx \sum_{j=1}^{M}\omega_j\, f(\xi_j) \tag{5.51}$$

of sufficiently high accuracy. The resulting algorithm, called Stieltjes procedure (see [G]), is more stable than the well known method of modified moments. In the kth step of this algorithm one needs the values $P_k^{\alpha,\beta}(\xi_j)$, $j = 1,\dots,M$, if (5.51) is used to approximate $I_{\alpha,\beta}$. More precisely, $\omega_j\big(P_k^{\alpha,\beta}(\xi_j)\big)^2$ appears in the quadrature rule (5.51) applied to $\big(P_k^{\alpha,\beta}\big)^2$ and $x\big(P_k^{\alpha,\beta}\big)^2$, respectively. In order to reduce the number of operations we determine the values $q_{k,j} := \sqrt{\omega_j}\,P_k^{\alpha,\beta}(\xi_j)$ instead of $P_k^{\alpha,\beta}(\xi_j)$. So we obtain the following algorithm for the approximate determination of the recurrence coefficients $\alpha_k^{\alpha,\beta}$, $\beta_k^{\alpha,\beta}$, $k = 0,\dots,n-1$:

$$\begin{aligned}
&j = 1,\dots,M:\ q_{-1,j} := 0\,,\ q_{0,j} := \sqrt{\omega_j}\,,\\
&a_0 = \sum_{j=1}^{M}\omega_j\,,\ b_0 = \sum_{j=1}^{M}\omega_j\,\xi_j\,,\ \alpha_0^{\alpha,\beta} = \frac{b_0}{a_0}\,,\ \beta_0^{\alpha,\beta} = \sqrt{a_0}\,,\\
&k = 1,\dots,n-1:\\
&\qquad j = 1,\dots,M:\ q_{kj} = \big(\xi_j - \alpha_{k-1}^{\alpha,\beta}\big)q_{k-1,j} - \big(\beta_{k-1}^{\alpha,\beta}\big)^2 q_{k-2,j}\,,\\
&\qquad a_k = \sum_{j=1}^{M}(q_{kj})^2\,,\ b_k = \sum_{j=1}^{M}(q_{kj})^2\xi_j\,,\ \alpha_k^{\alpha,\beta} = \frac{b_k}{a_k}\,,\ \beta_k^{\alpha,\beta} = \sqrt{\frac{a_k}{a_{k-1}}}\,.
\end{aligned}$$

A more stable implementation of this algorithm can be obtained if one takes into account that

$$a_k \approx I_{\alpha,\beta}\big((P_k^{\alpha,\beta})^2\big) = \frac{1}{\big(c_k^{\alpha,\beta}\big)^2} \sim \frac{1}{2^{2k}} \tag{5.52}$$

(see [F, Table V.A, p.246]). Indeed, (5.52) shows that it is much better to replace q_{kj} by $\widetilde{q}_{kj} = 2^k q_{kj}$ in the definition of the numbers a_k and b_k appearing in the kth step of the Stieltjes procedure. In this way we obtain numbers $\widetilde{a}_k \sim 1$ and $\widetilde{b}_k \le c$. Clearly, now we have to make the following modifications in the above algorithm:

$$\widetilde{q}_{kj} = 2\big(\xi_j - \alpha_{k-1}^{\alpha,\beta}\big)\widetilde{q}_{k-1,j} - 4\big(\beta_{k-1}^{\alpha,\beta}\big)^2\widetilde{q}_{k-2,j}\,,\quad \alpha_k^{\alpha,\beta} = \frac{\widetilde{b}_k}{\widetilde{a}_k}\,,\quad \beta_k^{\alpha,\beta} = \frac{1}{2}\sqrt{\frac{\widetilde{a}_k}{\widetilde{a}_{k-1}}}\,.$$

One should also take into account that

$$\lim_{k\to\infty}\alpha_k^{\alpha,\beta} = 0 \quad\text{and}\quad \lim_{k\to\infty}\beta_k^{\alpha,\beta} = \frac{1}{2}$$

(see [N, Lemmas 7.7 and 7.8]), since this gives the possibility to stop the Stieltjes procedure as soon as repeated computation does not yield new values for the recurrence coefficients in the sense that the differences to 0 and 0.5, respectively, are less than some prescribed error bound ε.

We recall that, for our quadrature method, the recurrence coefficients with respect to both pairs (α,β) and $(-\alpha,-\beta)$ are needed. The following lemma shows that the complete Stieltjes procedure only has to be applied to one of these pairs.

Lemma 5.21 $\alpha_k^{-\alpha,-\beta}=\alpha^{\alpha,\beta}_{k+\widetilde{k}+\widetilde{l}-1}$, $k\in\mathbb{N}$, $\beta_k^{-\alpha,-\beta}=\beta^{\alpha,\beta}_{k+\widetilde{k}+\widetilde{l}-1}$, $k\ge\max\left\{1,\,2-\widetilde{k}-\widetilde{l}\right\}$.

Now we know how to determine $t_k=x^{\alpha,\beta}_{n,k}$, $k=1,\dots,n$, and $y_k=x^{-\alpha,-\beta}_{m,k}$, $k=1,\dots,m$. The additional knots are defined by (5.19),

$$t_0=y_0=\frac{\min\{t_1,\,y_1\}-1}{2}\,,\quad t_{n+1}=y_{m+1}=\frac{\max\{t_n,\,y_m\}+1}{2}\,.$$

The weights $\sigma_k=\lambda^{\alpha,\beta}_{n,k}$, $k=1,\dots,n$, and $\mu_k=\lambda^{-\alpha,-\beta}_{m,k}$, $k=1,\dots,m$, of the Gaussian quadrature rules can be computed with the help of the following well known formula (see, e.g., [F, (I.4.7)])

$$\lambda^{\alpha,\beta}_{n,k}=\left(\sum_{i=0}^{n-1}\left[p_i^{\alpha,\beta}(x^{\alpha,\beta}_{n,k})\right]^2\right)^{-1}.$$

Since $\left(p_0^{\alpha,\beta},\dots,p_{n-1}^{\alpha,\beta}\right)\left(x^{\alpha,\beta}_{n,k}\right)$ is an eigenvector of the matrix (5.49) with respect to the eigenvalue $x^{\alpha,\beta}_{n,k}$, it turns out that solving the eigenvalue problem for (5.49) yields both the knots and the weights of the Gaussian quadrature rule $Q_n^{\alpha,\beta}$. Alternatively, one can use the recurrence formula (5.50) to obtain the values $p_i^{\alpha,\beta}\left(x^{\alpha,\beta}_{n,k}\right)$.

Remark 5.22 *Based on good numerical experiences we suggest a quadrature rule (5.51) which can be obtained as follows. Let*

$$\frac{1}{\pi}\int_{-1}^{1}f(x)\,v^{\alpha,\beta}(x)\,dx\approx\sum_{j=1}^{M}\lambda_j\,f(\xi_j)$$

be the Gaussian quadrature rule with respect to the Jacobi weight $v^{\alpha,\beta}$, determined by solving the corresponding eigenvalue problem for the tridiagonal matrix containing the well known recurrence coefficients of the normalized Jacobi polynomials (see, e.g., [Na]). Then the Quadrature rule

$$\sum_{j=1}^{M}\omega_j\,f(\xi_j)=\sum_{j=1}^{M}\lambda_j\left[\mathrm{e}^{-2}(1-\xi_j)^{1-\xi_j}(1+\xi_j)^{1+\xi_j}\right]^{\frac{(\operatorname{sign}\alpha+\operatorname{sign}\beta)/2-\alpha-\beta}{2}}\cos(\ell_{\alpha,\beta}(\xi_j))\,f(\xi_j)$$

(compare the definition of $\sigma_{\alpha,\beta}$) seems to be well appropriate to be used in the Stieltjes procedure. For example, for several values of α and β numerical tests have shown that, for the more or less exact determination (error $\lesssim 10^{-13}$; computations with double precision) of the first 700 recurrence coefficients, $M=3000$ is sufficient.

The following lemma shows how the remaining parameters $L_{j,k}$ and $P_{k,i}$ can be computed with the help of the values

$$p_k:=p_n^{\alpha,\beta}(y_k)\,,\ \ k=1-s,\dots,m+r\,,\quad\text{and}\quad q_k:=p_m^{-\alpha,-\beta}(t_k)\,,\ \ k=1-s,\dots,n+r\,,$$

which can be determined using the recurrence formula (5.50).

Lemma 5.23

$$L_{j,k} = \operatorname{sign}(\alpha)\, \frac{\mu_j\, q_k}{p_j\,(t_k - y_j)} \prod_{i\in\{1-s,m+r\}\cap\{0,m+1\}} \frac{t_k - y_i}{y_j - y_i} \prod_{i=1}^{N} \frac{y_j - y_{k(i)}}{t_k - y_{k(i)}}\,,$$
$$j \in \{1,\ldots,m\}\setminus\{k(1),\ldots,k(N)\},\ k = 1,\ldots,n\,,$$

$$L_{0,k} = \frac{q_k}{q_0} \prod_{i\in\{m+r\}\cap\{m+1\}} \frac{t_k - y_i}{y_0 - y_i} \prod_{i=1}^{N} \frac{y_0 - y_{k(i)}}{t_k - y_{k(i)}}\,, \quad k = 1,\ldots,n \quad (\textit{if } s = 1)\,,$$

$$L_{m+1,k} = \frac{q_k}{q_{n+1}} \prod_{i\in\{1-s\}\cap\{0\}} \frac{t_k - y_i}{y_{m+1} - y_i} \prod_{i=1}^{N} \frac{y_{m+1} - y_{k(i)}}{t_k - y_{k(i)}}\,, \quad k = 1,\ldots,n \quad (\textit{if } r = 1)\,,$$

$$L_{k,n+i} = -\frac{p_{k(i)}\,\mu_k}{p_k\,\mu_{k(i)}} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{y_k - y_{k(j)}}{y_{k(i)} - y_{k(j)}} \prod_{j\in\{1-s,m+r\}\cap\{0,m+1\}} \frac{y_{k(i)} - y_j}{y_k - y_j}\,,$$
$$k \in \{1,\ldots,m\}\setminus\{k(1),\ldots,k(N)\},\ i = 1,\ldots,N\,,$$

$$L_{0,n+i} = \operatorname{sign}(\alpha)\, \frac{p_{k(i)}(y_0 - y_{k(i)})}{q_0\,\mu_{k(i)}} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{y_0 - y_{k(j)}}{y_{k(i)} - y_{k(j)}} \prod_{j\in\{m+r\}\cap\{m+1\}} \frac{y_{k(i)} - y_j}{y_0 - y_j}\,,$$
$$i = 1,\ldots,N \quad (\textit{if } s = 1)\,,$$

$$L_{m+1,n+i} = \operatorname{sign}(\alpha)\, \frac{p_{k(i)}(y_{m+1} - y_{k(i)})}{q_{n+1}\,\mu_{k(i)}} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{y_{m+1} - y_{k(j)}}{y_{k(i)} - y_{k(j)}} \prod_{j\in\{1-s\}\cap\{0\}} \frac{y_{k(i)} - y_j}{y_{m+1} - y_j}\,,$$
$$i = 1,\ldots,N \quad (\textit{if } r = 1)\,,$$

$$P_{k,i} = -\frac{q_{m(i)}\,\sigma_k}{q_k\,\sigma_{m(i)}} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{t_k - t_{m(j)}}{t_{m(i)} - t_{m(j)}} \prod_{j\in\{1-s,n+r\}\cap\{0,n+1\}} \frac{t_{m(i)} - t_j}{t_k - t_j}\,,$$
$$k \in \{1,\ldots,n\}\setminus\{m(1),\ldots,m(N)\},\ i = 1,\ldots,N\,,$$

$$P_{0,i} = \operatorname{sign}(\alpha)\, \frac{q_{m(i)}(t_{m(i)} - t_0)}{p_0\,\sigma_{m(i)}} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{t_0 - t_{m(j)}}{t_{m(i)} - t_{m(j)}} \prod_{j\in\{n+r\}\cap\{n+1\}} \frac{t_{m(i)} - t_j}{t_0 - t_j}\,,$$
$$i = 1,\ldots,N \quad (\textit{if } s = 1)\,,$$

$$P_{n+1,i} = \operatorname{sign}(\alpha)\, \frac{q_{m(i)}(t_{m(i)} - t_{n+1})}{p_{m+1}\,\sigma_{m(i)}} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{t_{n+1} - t_{m(j)}}{t_{m(i)} - t_{m(j)}} \prod_{j\in\{1-s\}\cap\{0\}} \frac{t_{m(i)} - t_j}{t_{n+1} - t_j}\,,$$
$$i = 1,\ldots,N \quad (\textit{if } r = 1)\,.$$

Remark 5.24 *The computation of the recurrence coefficients is the most expensive part of the preprocessing, since this must be done with very high accuracy in order to obtain the correct values $A_{k,j}$ and G_k in (5.48). But if one has once computed the first n_0 recurrence* *coefficients, then one can use them for different values of n, supposed that $n \le n_0$. For this reason let us consider the computational effort needed for the quadrature method under the assumption that the recurrence coefficients are already computed. Then we obtain a complexity of $O(n^3)$ operations if Gaussian eliminations are used to solve the system (5.48). Clearly, because of this high complexity our method cannot be used for very large values of n. On the other hand, in the following examples good results are already obtained for more or less small values of n (e.g., $n = 500$). We mention that the coefficient matrix of the system (5.48) has a certain structure if $k(x,t) \equiv 0$. This gives some hope that the Amosov idea (see [BHS1]) can be used to construct a fast algorithm (with complexity $O(n^2)$ or even better) based on the quadrature method, for which the same error estimate can be proved under more restrictive smoothness assumptions on $k(x,t)$. It is planned to study this subject in a forthcoming paper.*

5.5 Numerical examples

In all of the following examples we consider equations (5.1) with $\varkappa_0 \in \{0,1,2\}$ and we determine u in such a way that the main part equation $A\phi = g$ possesses at most one solution in $\mathbf{H}_u$ for all right hand sides $g \in \bigcup_{\gamma>0} \mathbf{C}_u^{\gamma,0}$ (see Proposition 5.5). Moreover, we give concrete right hand sides g for which a solution $\phi^* \in \mathbf{H}_u$ of (5.1) is known and we hope that ϕ^* is the only $\mathbf{H}_u$–solution of (5.1) and that the corresponding function $f^* = \sigma_{\alpha,\beta}^{-1}\phi^*$ is the only $\mathbf{H}_{u(\alpha,\beta)}$–solution of (5.14). To compare the theoretical convergence order

$$\left\|\phi^* - \phi_n^*\right\|_{\widetilde{u}} = O\big(n^{\varepsilon-\widetilde{\gamma}}\big)\,, \quad \widetilde{u}(x) = v^{\tau_{N+1},\tau_0}(x)\prod_{j=1}^{N}|x - x_j|\,,$$

obtained by Corollary 5.18 ($\phi_n^* = \sigma_{\alpha,\beta}\, f_n^*$; see Remark 5.16) with the practical results, we have computed the vectors

$$\big(f_n^*\big(x_{n,k}^{\alpha,\beta}\big)\big)_{k\in\{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(n)\}}$$

(as described in the preceding section) for different values of n. Instead of $\|\phi^* - \phi_n^*\|_{\widetilde{u}}$ (which cannot be computed exactly) we give the values

$$\left\|\phi^* - \phi_n^*\right\|_u^{\sim} := \max_{i=1,\dots,\widetilde{n}}\left|(\widetilde{u}\,\phi^*)(\xi_i) - (\widetilde{u}\,\sigma_{\alpha,\beta}S_n^*)(\xi_i)\right|,$$

where $\{\xi_i\}_{i=1}^{\widetilde{n}} = \{-1 + 0.0001\,j\}_{j=0}^{20000} \cup \big\{x_{n,k}^{\alpha,\beta}\big\}_{k\in\{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(N)\}}$ and S_n^* is a cubic $\mathbf{C}^2$–spline interpolating f_n^* at the points $x_{n,k}^{\alpha,\beta}$, $k \in \{1-s,\dots,n+r\}\setminus\{m(1),\dots,m(N)\}$ (obtained by the MATLAB-function *spline*). We use S_n^* instead of the polynomial f_n^*, since S_n^* is smooth enough for our purposes and spline interpolation is less expensive (and more stable) than polynomial interpolation (which would be necessary to compute the values $f_n^*(\xi_i)$). We omit a theoretical investigation of the error $\|f_n^* - S_n^*\|_{\widetilde{u}(\alpha,\beta)}$, since we do not expect that this error has big influence in our examples in which only more or less simple functions f^* are considered.

Example 5.25 *Let $a(x)=\sqrt{1-x^2}$ and $b(x)=-x$. We have*

$$a,b\in\mathbf{C}^{1,0}\,,\quad \varkappa_0=2\,,\quad \alpha_0=\beta_0=-\frac{1}{2}\,.$$

In view of Proposition 5.5, it is reasonable to choose τ_0 and τ_{N+1} less than $1/2$. One can easily check that $\phi^*(t)=2(1-t^2)$ is the solution of $(A+K)\phi=g$ for

$$k(x,t)=1+\sqrt{\frac{1-x^2}{1-t^2}}\,,\quad g(x)=2(1-x^2)^{3/2}+\sqrt{1-x^2}+\frac{4x^2}{\pi}-\frac{2x(1-x^2)}{\pi}\ln\frac{1-x}{1+x}\,.$$

For every fixed $\varepsilon\in(0,1/2)$ and for $w(t)=(1-t^2)^{1-\varepsilon}$ we have

$$\sup_{t\in(-1,1)}\|k(\,.\,,t)\,w(t)\|_{1,0}<\infty\,,\qquad \sup_{x\in(-1,1)}\|k(x,\,.\,)\|_{w,1-2\varepsilon,0}<\infty\,,\qquad g\in\mathbf{C}^{1,0}$$

(see Lemma 2.27 and Corollary 1.18, (i)). Thus, by Theorem 5.14, we expect that

$$\|\phi^*-\phi_n^*\|\le c\,n^{\varepsilon-1}\quad\textit{for every fixed }\varepsilon>0\,.$$

Since $a'\notin\mathbf{C}[-1,1]$, we cannot set $d=0$ in the definition of $\mathcal{H}_n$ if Theorem 5.14 should be applicable. But the following table shows that even better numerical results are obtained if we set $d=0$. We conjecture that this is the case since a is smooth inside $(-1,1)$.

The functions a and b are chosen in such a way that also spectral methods can be implemented easily. We take the numerical results for a corresponding quadrature method from [JL1, Example 7.1]. It seems that the order of uniform convergence is $O(n^{-2})$ for our quadrature method with $d=0$ and the quadrature method from [JL1], although in both cases only $O(n^{\varepsilon-1})$ can be expected theoretically.

	quadrature method		quadrature method, $d=0$		method from [JL1]	
n	$\|\phi^*-\phi_n^*\|^\sim$	$n\|\phi^*-\phi_n^*\|^\sim$	$\|\phi^*-\phi_n^*\|^\sim$	$n^2\|\phi^*-\phi_n^*\|^\sim$	$\|\phi^*-\phi_n^*\|$	$n^2\|\phi^*-\phi_n^*\|$
10	0.122441	1.22	1.482 e-2	1.48	0.01380	1.38
20	0.065459	1.31	4.222 e-3	1.69	0.00366	1.46
30	0.044823	1.34	1.961 e-3	1.76	0.00166	1.49
40	0.034074	1.36	1.127 e-3	1.80	0.00094	1.50
50	0.027486	1.37	7.322 e-4	1.83	0.00012	0.30
60	0.023032	1.38	5.135 e-4	1.85	0.00008	0.29
70	0.019821	1.39	3.726 e-4	1.83	0.00006	0.29
80	0.017395	1.39	2.914 e-4	1.86	0.00004	0.26
90	0.015499	1.39	2.313 e-4	1.87	0.00004	0.32
100	0.013975	1.40	1.846 e-4	1.85	0.00002	0.20
200	0.007048	1.41	2.295 e-5	0.92		
300	0.004712	1.41	9.247 e-6	0.83		
400	0.003539	1.42	5.768 e-6	0.92		
500	0.002834	1.42	2.749 e-6	0.69		
600	0.002363	1.42	2.095 e-6	0.75		
700	0.002026	1.42	6.368 e-7	0.31		

Example 5.26 *We take a and b as in Example 5.25. The function $\phi^*(t) = 2(1-t^2)$ is the solution of $(A+K)\phi = g$ for*

$$k(x,t) = 1 + t\sqrt{|x|(1-t^2)}\,, \quad g(x) = 2(1-x^2)^{3/2} + \frac{4x^2}{\pi} - \frac{2x(1-x^2)}{\pi}\ln\frac{1-x}{1+x}\,.$$

We have $g \in \mathbf{C}^{1,0}$, $k(\,.\,,t) \in \mathbf{C}^{1/2,0}$ uniformly with respect to t, and $k(x,\,.\,) \in \mathbf{C}^{1,0}$ uniformly with respect to x. Thus, theoretically we can expect

$$\|\phi^* - \phi_n^*\| \le c\,n^{\varepsilon - 1/2} \quad \textit{for every fixed } \varepsilon > 0\,.$$

The same theoretical result holds for the spectral method which is considered in [CJLM, Example 3]. The numerical results are much better.

	quadrature method		quadrature method, $d=0$		method from [CJLM]	
n	$\|\phi^*-\phi_n^*\|^\sim$	$n\|\phi^*-\phi_n^*\|^\sim$	$\|\phi^*-\phi_n^*\|^\sim$	$n^2\|\phi^*-\phi_n^*\|^\sim$	$\|\phi^*-\phi_n^*\|$	$n^2\|\phi^*-\phi_n^*\|$
10	0.167472	1.67	1.481 e-2	1.48	0.0138	1.38
20	0.090887	1.82	4.217 e-3	1.69	0.0036	1.44
30	0.062500	1.88	1.960 e-3	1.76	0.0016	1.44
40	0.047618	1.90	1.127 e-3	1.80	0.0010	1.60
50	0.038461	1.92	7.319 e-4	1.83	0.0006	1.50
60	0.032258	1.94	5.133 e-4	1.85	0.0004	1.44
70	0.027778	1.94	3.725 e-4	1.83	0.0004	1.96
80	0.024390	1.95	2.913 e-4	1.86	0.0002	1.28
100	0.019608	1.96	1.845 e-4	1.85		
200	0.009901	1.98	2.293 e-5	0.92		
300	0.006623	1.99	9.244 e-6	0.83		
400	0.004975	1.99	5.767 e-6	0.92		
500	0.003984	1.99	2.749 e-6	0.69		
600	0.003322	1.99	2.094 e-6	0.75		
700	0.002849	1.99	6.365 e-7	0.31		

Example 5.27 *In [W, Example 8.2.(A)] the equation $\big(\mathrm{sign}(x)I + xS\big)\phi = x + \frac{x^2}{\pi}\ln\frac{1-x^2}{x^2}$ is considered and numerical results for a non-spectral collocation method based on the zeros of the Chebychev polynomials of the second kind are presented. More precisely, approximative solutions $\phi_n \in \varphi\Pi_n$ are determined and the errors $\|(\phi_n - \phi_n^{\mathrm{best}})\,\varphi^{-1/2}\|_{\mathbf{L}^2(-1,1)}$ are computed, where ϕ_n^{best} is that element of $\varphi\Pi_n$ with the smallest distance in the norm $\|\,.\,\varphi^{-1/2}\|_{\mathbf{L}^2}$ to the solution*

$$\phi^*(t) = |t|$$

of the above equation. We multiply this equation from the left by $\mathrm{sign}(x)$ to obtain an equivalent equation $(aI + bS)\phi = g$ with $a, b \in \mathbf{C}$, which can be handled by our quadrature method if we write $bS = SbI + K$, $K = bS - SbI$. Thus, we consider

$$a(x) = 1\,, \quad b(x) = |x|\,, \quad k(x,t) = \frac{|x| - |t|}{t - x}\,, \quad g(x) = |x|\left(1 + \frac{x}{\pi}\ln\frac{1-x^2}{x^2}\right)\,.$$

We have $\varkappa_0 = 1$, $\alpha_0 = -1/4$, $\beta_0 = -3/4$. *If we take into account that* g *has logarithmic singularities in* ± 1, *then Proposition 5.5 suggest to take* u *with* $\tau_0 \geq 3/4$, $0 < \tau_{N+1} < 1/4$ *or* $0 < \tau_0 < 3/4$, $\tau_{N+1} \geq 1/4$. *We choose*

$$N = 0 \quad \textit{and} \quad \tau_0 = \tau_1 = \frac{1}{2}, \quad \textit{i.e.,} \quad u = \varphi .$$

In view of assertion (i) of Corollary 1.18, we have $g \in \mathbf{C}^{1,0}_{\varphi}$. *Moreover,* $a, b, h(x, \,.\,), h(\,.\,, t) \in \mathbf{C}^{1,0}$, *where* $h(x,t) = (t-x)\,k(x,t)$. *By Theorem 5.14, we expect that*

$$\|\phi^* - \phi_n^*\|_{\varphi} \leq c\, n^{\varepsilon - 1} \quad \textit{for every fixed } \varepsilon > 0 .$$

The computed errors seem to converge with order $O(n^{-1})$. *We compare these errors with the results obtained by the collocation method from [W], although* $\|\,.\,\|_{\varphi}$ *and* $\|\,.\,\varphi^{-1/2}\|_{\mathbf{L}^2}$ *are not comparable (i.e., none of these norms can be estimated by a multiple of the other). We also mention that the comparison of the errors obtained for the same values of* n *is not really justified, since we need* $O(n^3)$ *operations for the quadrature method, while the method from [W] works with* $O(n^2)$ *operations.*

	quadrature method		collocation method from [W]	
n	$\|\phi^* - \phi_n^*\|_{\varphi}^{\sim}$	$n\|\phi^* - \phi_n^*\|_{\varphi}^{\sim}$	$\|(\phi_n^{\text{best}} - \phi_n^*)\,\varphi^{-1/2}\|_{\mathbf{L}^2}$	$n^{1/2}\|(\phi_n^{\text{best}} - \phi_n^*)\,\varphi^{-1/2}\|_{\mathbf{L}^2}$
20	0.048837	0.98	0.144497	0.65
40	0.024977	1.00	0.105004	0.66
60	0.016769	1.01	0.086535	0.67
80	0.012619	1.01	0.075295	0.67
100	0.010116	1.01	0.067538	0.68
500	0.002036	1.02	0.030405	0.68

Example 5.28 *Let us compare again our quadrature method with the collocation method from [W] (see Example 5.27), where we consider the following equation taken from [W, Example 8.2.(B)]:* $(aI + SbI + K)\phi = g$ *with*

$$a = 2\,, \quad b(x) = 1 - x^2\,, \quad k(x,t) = x + t\,, \quad g(x) = (1 - x^2)\left(2 - \frac{2x}{\pi} + \frac{1 - x^2}{\pi}\ln\frac{1 - x}{1 + x}\right).$$

One can easily check that $\phi^*(t) = 1 - t^2$ *is a solution. We have* $\varkappa_0 = \alpha_0 = \beta_0 = 0$ *and, by Proposition 5.5, it seems that any choice of* $u = v^{\tau_0,\tau_1}$ *with* $0 < \tau_0, \tau_1 < 1$ *is allowed. By Theorem 5.14, we expect*

$$\|\phi^* - \phi_n^*\|_{\varphi} \leq c\, n^{\varepsilon - 1} \quad \textit{for every fixed } \varepsilon > 0\,,$$

if α *and* β *are taken from* $(-1/2, 0)$. *(Note that* $b(-1) = b(1) = 0$.*) In the following table we present the numerical results obtained for* $\alpha = \beta = -1/4$ *and we compare them with that obtained by the collocation method from [W] (although this is not really justified, since the norms are not comparable and the complexities of the methods are different). We have taken* $d = 0$ *in the definition of* $\mathcal{H}_n$, *since the numerical results are better than that obtained with the theoretically correct definition of* $\mathcal{H}_n$.

	quadrature method, $d=0$		collocation method from [W]	
n	$\Vert\phi^*-\phi_n^*\Vert_\varphi^\sim$	$n^3\Vert\phi^*-\phi_n^*\Vert_\varphi^\sim$	$\Vert(\phi_n^{\text{best}}-\phi_n^*)\,\varphi^{-1/2}\Vert_{\mathbf{L}^2}$	$n^{5/2}\Vert(\phi_n^{\text{best}}-\phi_n^*)\,\varphi^{-1/2}\Vert_{\mathbf{L}^2}$
20	1.957 e-4	1.57	4.078 e-4	0.73
40	2.259 e-5	1.45	8.133 e-5	0.82
60	6.535 e-6	1.41	3.080 e-5	0.86
80	2.729 e-6	1.40	1.533 e-5	0.88
100	1.387 e-6	1.39	8.891 e-6	0.89
500	1.021 e-8	1.28	1.660 e-7	0.93

Example 5.29 *We consider the equation $(aI+SbI+K)\phi=g$ with*

$$a=2,\;\; b(x)=(1-x^2)^{3.1},\;\; k(x,t)=\frac{(1-x^2)^{3.1}-(1-t^2)^{3.1}}{t-x}+\pi(1+x\sin t)(1-x^2)^{1.5},$$

$$g(x)=(1-x^2)\left(2+\frac{4}{3}\sqrt{1-x^2}+(1-x^2)^{2.1}\left(-\frac{2x}{\pi}+\frac{1-x^2}{\pi}\ln\frac{1-x}{1+x}\right)\right)$$

(see [W, Example 10.1.(K2)]). We have $\phi^*(t)=1-t^2$ and $\varkappa_0=\alpha_0=\beta_0=0$. As in Example 5.28 we expect

$$\Vert\phi^*-\phi_n^*\Vert_\varphi\le c\,n^{\varepsilon-1}\quad \textit{for every fixed } \varepsilon>0\,,$$

if α and β are taken from $(-1/2,\,0)$. The following results are obtained for $\alpha=\beta=-1/4$ and with d $=0$ in the definition of $\mathcal{H}_n$ (since this yields better numerical results than in the case of the theoretically correct defined $\mathcal{H}_n$). We compare these results with that obtained by the (non-spectral) fixed point iteration method considered in [W, Section 10.2].

	quadrature method, $d=0$		iteration method from [W]	
n	$\Vert\phi^*-\phi_n^*\Vert_\varphi^\sim$	$n^{3.25}\Vert\phi^*-\phi_n^*\Vert_\varphi^\sim$	$\Vert(\phi_n^{\text{best}}-\phi_n^*)\,\varphi^{-1/2}\Vert_{\mathbf{L}^2}$	$n^{2.5}\Vert(\phi_n^{\text{best}}-\phi_n^*)\,\varphi^{-1/2}\Vert_{\mathbf{L}^2}$
100	1.386 e-6	4.38	1.345 e-5	1.35
200	1.679 e-7	5.05	2.406 e-6	1.36
400	1.058 e-8	3.03	4.278 e-7	1.37
800	2.358 e-10	0.64	7.58 e-8	1.37

Example 5.30 *The function $\phi^*(t)=\operatorname{sign}(t)$ is a solution of $(aI+SbI+K)\phi=g$ for*

$$a(x)=1-|x|,\;\; b(x)=|x|,\;\; k(x,t)=\operatorname{sign}(x)\,|t|,\;\; g(x)=\operatorname{sign}(x)-x+\frac{2}{\pi}+\frac{x}{\pi}\ln\frac{1-x}{1+x}\,.$$

We have $\varkappa_0=1$ and $\alpha_0=\beta_0=-1/2$. In view of Proposition 5.5, we conjecture that ϕ^* is the only $\mathbf{H}_u$–solution if

$$u(x)=(1+x)^{\tau_0}|x|^{\tau_1}(1-x)^{\tau_2}\;\;\textit{with}\;\; 0<\tau_i<1\;\;\textit{and}\;\;\min\{\tau_0,\tau_2\}<\frac{1}{2}\,.$$

By assertion (iii) of Corollary 1.18, $a,b,k(x,\,.\,)\in\mathbf{C}^{1,0}$ and $\operatorname{sign}(\,.\,)\in\mathbf{C}_u^{\tau_1,0}$. Together with assertion (i) of Corollary 1.18 and assertion (vii) of Theorem 1.11 we obtain $g,k(\,.\,,t)\in$

$\mathbf{C}_u^{\tau_1,0} \cup \mathbf{C}_\varphi^{1,0} \subseteq \mathbf{C}_u^{\min\{\tau_1,2\tau_0,2\tau_2\},0}$. *To obtain a theoretical convergence order which is close to* $O(n^{-1})$ *we consider the weights*

$$\widetilde{u}_1(x) = (1-x^2)^{0.45}|x| \quad \textit{and} \quad \widetilde{u}_2(x) = (1+x)^{1/2}|x|(1-x)^{0.45}\,.$$

In view of Corollary 5.18, we expect that, in both cases,

$$\|\phi^* - \phi_n^*\|_{\widetilde{u}_i} \le c\, n^{\varepsilon - 0.9} \quad \textit{for every fixed } \varepsilon > 0\,.$$

The numerical results seem to show that this is even true with $c\,n^{-1}$ *on the right hand side.*

	quadrature method, $\tau_0 = \tau_2 = 0.45$		quadrature method, $\tau_0 = 0.5$, $\tau_2 = 0.45$	
n	$\|\phi^* - \phi_n^*\|_{\widetilde{u}_1}$	$n\|\phi^* - \phi_n^*\|_{\widetilde{u}_1}$	$\|\phi^* - \phi_n^*\|_{\widetilde{u}_2}$	$n\|\phi^* - \phi_n^*\|_{\widetilde{u}_2}$
100	0.021583	2.16	0.143861	14.39
200	0.011297	2.26	0.062660	12.53
300	0.007552	2.27	0.039960	11.99
400	0.005669	2.27	0.029306	11.72
500	0.004536	2.27	0.023127	11.56
600	0.003781	2.27	0.019096	11.46
700	0.003242	2.27	0.016260	11.38

5.6 Proofs

5.6.1 Proofs of Propositions 5.1, 5.3, 5.4, and 5.5

Proof of Proposition 5.1. From assertion (v) of Corollary 1.18 and assertion (i) of Proposition 1.19 it follows that the conditions $g \in \mathbf{H}_u$ and $(gu)(x_j) = 0$ for all j with $\tau_j \neq 0$ are equivalent to the existence of a pair $(\gamma,\delta) \in (0,\infty)\times\mathbb{R}$ such that $g \in \mathbf{C}_u^{\gamma,\delta}$. More precisely, for every $\eta > 0$ there exists some $\gamma(\eta,u) > 0$ such that

$$\mathbf{H}_u^\eta := \big\{g : \, gu \in \mathbf{H}^\eta([-1,1]) \text{ and } (gu)(x_j) = 0 \text{ for all } j \text{ with } \tau_j \neq 0\big\}\,,$$

endowed with $\|g\| = \|gu\|_{\mathbf{H}^\eta}$, is continuously embedded into $\mathbf{C}_u^{\gamma,0}$ and for every pair (γ,δ) there exists an $\eta(\gamma,\delta,u) > 0$ such that $\mathbf{C}_u^{\gamma,\delta}$ is continuously embedded into $\mathbf{H}_u^\eta$. This is also true if $u = v^{\tau_0,\tau_1}$ is a Jacobi weight with exponents $\tau_i \in (-1,1)$. Indeed, if $p \in \{v^{0,0}, v^{1,0}, v^{0,1}, v^{1,1}\}$ such that the exponents of pu lie in $[0,1)$, then $f \in \mathbf{C}_u$ iff $f/p \in \mathbf{C}_{up}$ and the identity

$$E_{n+\deg p}^u(f) = \inf_{p_n \in \Pi_n} \|(f - p_n p)u\| = E_n^{pu}(f/p)$$

shows that $f \in \mathbf{C}_u^{\gamma,\delta}$ if and only if $f/p \in \mathbf{C}_{up}^{\gamma,\delta}$ (with equivalent norms). Thus, the above stated connection between the spaces $\mathbf{H}_u^\eta$ and $\mathbf{C}_u^{\gamma,\delta}$ can be proved via the corresponding connection between $\mathbf{H}_{up}^\eta$ and $\mathbf{C}_{up}^{\gamma,\delta}$. Now we obtain that, for the function h_1 from (5.4), there exists of a pair (γ,δ) such that (5.7) is satisfied if and only if there exists an $\eta > 0$ such that

$$u^{-1}(\,.\,)\, h_1(\,.\,,t) \in \mathbf{H}_u^\eta \quad \text{uniformly with respect to } \; t \in [-1,1] \quad \text{and}$$

$$h_1(x,\,.\,)w^{-1}(\,.\,) \in \mathbf{H}_w^\eta \quad \text{uniformly with respect to } \; x \in [-1,1]\,.$$

By "uniformly" we mean that the norm can be estimated by a constant which does not depend on t and x, respectively. Particularly, $h_1(\,.\,,t) \in \mathbf{H}^\eta([-1,1])$ uniformly w.r.t. $t \in [-1,1]$ and $h_1(x,\,.\,) \in \mathbf{H}^\eta([-1,1])$ uniformly w.r.t. $x \in [-1,1]$, which is equivalent to $h_1 \in \mathbf{H}^\eta([-1,1]^2)$. A similar consideration yields the assertion for $h_2(x,t)$. In Lemma 3.9 we have seen that (even for arbitrary continuous weight functions u) the first supremum in (5.7) is already finite if the corresponding supremum over all t from a subset $D \subseteq [-1,1]$ of measure 2 is finite. An analogous assertion holds true for the second supremum in (5.7), since

$$\left\|h_1(x,\,.\,)w^{-1}(\,.\,)\right\|_{w,\gamma,\delta} \sim \left\|h_1(x,\,.\,)(wp)^{-1}(\,.\,)\right\|_{pw,\gamma,\delta}$$

with p from Remark 5.2. Clearly, the same is true for the suprema in (5.8). ∎

Proof of Proposition 5.3. Theorems 3.1 and 3.5 imply $K \in \mathcal{L}\big(\mathbf{C}_u, \mathbf{C}_u^{\gamma,\delta}\big)$. The first assertion of (5.9) follows from (2.22). ∎

Proof of Proposition 5.4. From (5.6) and (5.9) it follows

$$A\phi = g - K\phi \in \mathbf{C}_u^{\gamma,\delta}\,. \tag{5.53}$$

Particularly, $A\phi \in \mathbf{H}_u$ (Proposition 1.19) and, consequently, $A\phi \in v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1})$, where k and l are defined in (2.54). Now we obtain from (5.53) and Remark 2.14 that ϕ must be of the form

$$\phi = B_{k,l}(g - K\phi) + v^{k,l}\sigma_0\, p\,, \tag{5.54}$$

where $p \in \Pi$. In view of Lemma 2.18, this implies

$$\phi \in v^{k,l}\sigma_0\mathbf{C}^{\gamma,\delta-1}_{u(\alpha_0+k,\beta_0+l)}\,, \quad \text{where} \quad u(\alpha,\beta) = v^{\alpha,\beta}u\,.$$

Together with Proposition 1.6 and $v^{k,l}\sigma_0 \sim v^{\alpha_0+k,\beta_0+l}h$ we get

$$(\phi u)(x_j) = 0 \;\text{ for all } x_j \;(j \in \{0,\dots,N+1\}) \text{ which are zeros of } u(\alpha_0+k,\beta_0+l)\,.$$

To prove the second assertion of Proposition 5.4 we set

$$l = 1 \;\text{ and }\; k = 0 \quad (l = 0 \;\text{ and }\; k = 1)\,.$$

Then, our assumption on g means $g \in \mathbf{H}^{k,l}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1})$, while the assumption on k implies that the conditions on $k_1(x,t) = k(x,t)$ which are given before Proposition 5.1 are even satisfied with u replaced by $u(0,-\tau_0)$ $(u(-\tau_{N+1},0))$, i.e., τ_0 (τ_{N+1}) replaced by 0. In view of Proposition 5.1 this means that, for a certain $\widetilde{\gamma} > 0$,

$$\sup_{t\in[-1,1]} \left\|k(\,.\,,t)w(t)\right\|_{u(0,-\tau_0),\widetilde{\gamma},0} < \infty \quad \left(\sup_{t\in[-1,1]} \left\|k(\,.\,,t)w(t)\right\|_{u(-\tau_{N+1},0),\widetilde{\gamma},0} < \infty\right)\,.$$

Now, Theorem 3.1 shows that $K\phi \in \mathbf{C}^{\widetilde{\gamma},0}_{u(0,-\tau_0)}$ $(K\phi \in \mathbf{C}^{\widetilde{\gamma},0}_{u(-\tau_{N+1},0)})$ which yields, in view of assertion (i) of Proposition 1.19,

$$g - K\phi \in \mathbf{H}^{k,l}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1})\,.$$

Taking (5.54) and (2.32) into account, we conclude $\phi \in \mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1})$. ■

Proof of Proposition 5.5. Define k and l by (2.54). Then, $\varkappa = \varkappa_0 - k - l = 0$. If we take into account that $g \in \mathbf{H}_u$ (see Proposition 1.19) and that

$$\mathbf{H}_u \subseteq \mathbf{H}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1})\,, \tag{5.55}$$

then we see that, by Remark 2.14, (5.1) possesses the uniquely determined solution

$$\phi = B_{k,l}\, g$$

in the space $\mathbf{H}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1}) \cap v^{\alpha_0,\beta_0}\mathring{\mathbf{H}}^{k,l}_{\mathrm{loc}}(\{x_i\}_{i=0}^{N+1})$. In view of (5.55), we conclude that (5.1) cannot have more than one solution in $\mathbf{H}_u$. It remains to mention that, by Lemma 2.18,

$$B_{k,l}\, g \in v^{k,l}\sigma_0 \mathbf{C}^{\gamma,\delta-1}_{u(\alpha_0+k,\beta_0+l)}\,,$$

which implies $B_{k,l}\, g \in \mathbf{H}_u$ (see Corollary 1.18, (v)). ■

5.6.2 Proof of Lemma 5.13

Lemma 5.31 *Lemma 2.30 remains true if $\rho,\tau > -1$ and if "f vanishes in the zeros of $v^{\rho,\tau}$" is replaced by "f vanishes in the zeros of $v^{|\rho|,|\tau|}$".*

Proof. Let $p = 1$ if $\rho < 0$ and $p = 0$ if $\rho \geq 0$. Analogously, define $q \in \{0,1\}$ in dependence of τ. Lemma 2.30 applied to $\widetilde{\rho} = \rho + p$ and $\widetilde{\tau} = \tau + q$ yields $fv^{-\rho-p,-\tau-q} \in \mathbf{C}^{\gamma,\delta}_{\rho+p,\tau+q}$. Together with

$$\left\| \left(fv^{-\rho-p,-\tau-q} - p_n \right) v^{\rho+p,\tau+q} \right\| = \left\| \left(fv^{-\rho,-\tau} - v^{p,q}p_n \right) v^{\rho,\tau} \right\|, \quad p_n \in \Pi_n\,,$$

we obtain the assertion. ■

Proof of Lemma 5.13. Lemma 5.31 applied to $f = \widetilde{a}$, $f = v^{\gamma/4,0}\,\widetilde{a}$, $f = v^{0,\gamma/4}\,\widetilde{a}$, and $f = v^{\gamma/4,\gamma/4}\,\widetilde{a}$, respectively, yields the assertion for $v^{-\alpha,-\beta}\,\widetilde{a}$ instead of $r^{-2}\sigma^{-1}_{\alpha,\beta}\,\widetilde{a}$. (Here we took Proposition 1.12, assertion (i) of Corollary 1.18, and Lemma 2.27 into account.) It remains to mention that Lemma 2.28 can be applied to obtain the assertion for $r^{-2}\sigma^{-1}_{\alpha,\beta}\,\widetilde{a}$, since

$$\frac{v^{\alpha,\beta}(x)}{\left(r^2\sigma_{\alpha,\beta}\right)(x)} = \frac{1}{r^2(x)\cos(\ell_{\alpha,\beta}(x))}\left[\mathrm{e}^{-2}(1-x)^{1-x}(1+x)^{1+x} \right]^{\frac{\alpha+\beta-(\operatorname{sign}\alpha+\operatorname{sign}\beta)/2}{2}} \tag{5.56}$$

belongs to $\mathbf{C}^{\gamma,\min\{\delta+1,0\}}$ if $b(-1)b(1) \neq 0$ and to $\mathbf{C}^{\gamma/2,\min\{\delta+1,0\}}$ if $b(-1)b(1) = 0$, where we used Proposition 1.14 (remark that the last factor on the right hand side of (5.56) is an element of $\mathbf{W}(1,2)$) and again Proposition 1.12 and Corollary 1.18. ■

5.6.3 Proof of Theorem 5.14

We already know that $\widehat{\mathcal{A}}$ is a left $(\mathcal{B},\mathcal{C})$–regularizer of $\mathcal{A} \in \mathcal{L}\big(\mathbf{X}_1^{\mathcal{L}},\mathbf{Y}\big) \cap \mathcal{L}\big(\mathbf{X}_1^{\mathcal{L}^2},\mathbf{Y}_1^{\mathcal{L}}\big)$, $\mathcal{L} = \{a_n\} := \{\log_2(n+1)\}$, where

$$\mathbf{X} = \mathbf{C}_{u(\alpha,\beta)}\,, \quad \mathbf{Y} = \mathbf{C}_u\,, \quad \mathbf{X}_n = \Pi_{n+r+s-N}\,, \quad \mathbf{Y}_n = \Pi_{n(\alpha,\beta)+r+s-N}\,,$$

$$b_n \sim n^{\gamma}\ln^{\delta} n\,, \quad c_n \sim n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}}$$

for sufficiently large n. Moreover, before Theorem 5.14 we have mentioned that $\mathcal{K} \in \mathcal{L}\big(\mathbf{X},\mathbf{Y}_\infty^{\mathcal{L}\mathcal{B}}\big)$. Furthermore, Remark 5.7 shows that $(I+\mathcal{H}+\widehat{\mathcal{A}}\mathcal{K})f = 0$ has only the trivial solution in $\mathbf{C}_{u(\alpha,\beta)}$. Thus, in view of Theorem 4.10 it is sufficient to prove that

$$\|P_n\|_{\mathcal{L}(\mathbf{X})},\, \|L_n\|_{\mathcal{L}(\mathbf{Y})} \le c\, n^{\max_{i=1,\dots,N}(1-\tau_i)} \ln n \qquad (n \ge n_0) \tag{5.57}$$

holds true for P_n, L_n from Definition 5.11 and that, for $n \ge n_0$,

$$\inf_{h_n\in\ker L_n} \|(\mathcal{K}-K_n)f_n - h_n\|_{\mathbf{Y}} \;\le\; \frac{c}{n^{\gamma}\ln^{\delta+1} n}\,\|f_n\|_{\mathbf{X}}\,, \quad f_n \in \mathbf{X}_n\,, \tag{5.58}$$

$$\inf_{h_n\in\ker P_n} \|(\mathcal{H}-H_n)f_n - h_n\|_{\mathbf{X}} \;\le\; \frac{c}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}} n}\,\|f_n\|_{\mathbf{X}}\,, \quad f_n \in \mathbf{X}_n\,, \tag{5.59}$$

$(c \ne c(n,f_n))$, where

$$K_n = L_n\mathcal{K}_n\,, \quad H_n = P_n\mathcal{H}_n$$

with $\mathcal{K}_n$ and $\mathcal{H}_n$ defined by (5.26) and (5.27), respectively. (Remark that, in general, we cannot take $K_n = \mathcal{K}_n$ and $H_n = \mathcal{H}_n$, since the images of these operators may be not contained in $\mathbf{Y}$ and $\mathbf{X}$, respectively. Clearly, replacing $\mathcal{K}_n$ and $\mathcal{H}_n$ by $L_n\mathcal{K}_n$ and $P_n\mathcal{H}_n$ does not change our approximation method (5.25).) For $f \in \mathbf{C}_u$ the infimum $\inf_{h_n\in\ker L_n}\|f-h_n\|_u$ is taken for $h_n = f - u^{-1}S$, where S is the linear spline which interpolates uf on the set $\big\{x^{-\alpha,-\beta}_{n(\alpha,\beta),j}\big\}_{j=1-s}^{n(\alpha,\beta)+r} \setminus \big\{x^{-\alpha,-\beta}_{n(\alpha,\beta),k(i)}\big\}_{i=1}^{N}$ (r and s from (5.23)) and vanishes in ± 1. Consequently, (5.58) and, analogously, (5.59) means

$$\max_{\substack{j=1-s,\dots,n(\alpha,\beta)+r\\ j\notin\{k(1),\dots,k(N)\}}} |u\,(\mathcal{K}-\mathcal{K}_n)f_n|\big(x^{-\alpha,-\beta}_{n(\alpha,\beta),j}\big) \;\le\; \frac{c}{n^{\gamma}\ln^{\delta+1} n}\,\|f_n\|_{\mathbf{X}}\,, \quad f_n \in \mathbf{X}_n\,, \tag{5.60}$$

$$\max_{\substack{j=1-s,\dots,n+r\\ j\notin\{m(1),\dots,m(N)\}}} |v^{\alpha,\beta}u\,(\mathcal{H}-\mathcal{H}_n)f_n|\big(x^{\alpha,\beta}_{n,j}\big) \;\le\; \frac{c}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}} n}\,\|f_n\|_{\mathbf{X}}\,, \quad f_n \in \mathbf{X}_n\,. \tag{5.61}$$

Proof of (5.57). We prove the following more general assertion which holds for arbitrary power weights u of the form (2.19) and arbitrary $\alpha,\beta \in (-1,1)\setminus\{0\}$:

$$\sup_{n\ge 2} \frac{\big\|L^{\alpha,\beta}_{n,u}\big\|_{\mathcal{L}(\mathbf{C}_u)}}{n^{\max\{1-\tau_1,\dots,1-\tau_N\}}\ln n} < \infty \quad \text{if} \quad \begin{aligned} &\frac{\alpha}{2}+\frac{5}{4} \ge \tau_{N+1} + r(\alpha,\beta,u)\,,\\ &\frac{\beta}{2}+\frac{5}{4} \ge \tau_0 + s(\alpha,\beta,u)\,. \end{aligned} \tag{5.62}$$

We remark that the corresponding assumptions for P_n and L_n,

$$\begin{aligned} &\frac{\alpha}{2}+\frac{5}{4} \ge \tau_{N+1}+\alpha+r(\alpha,\beta,u(\alpha,\beta))\,,\\ &\frac{\beta}{2}+\frac{5}{4} \ge \tau_0+\beta+s(\alpha,\beta,u(\alpha,\beta))\,, \end{aligned} \quad \text{and} \quad \begin{aligned} &-\frac{\alpha}{2}+\frac{5}{4} \ge \tau_{N+1}+r(-\alpha,-\beta,u)\,,\\ &-\frac{\beta}{2}+\frac{5}{4} \ge \tau_0+s(-\alpha,-\beta,u)\,, \end{aligned}$$

coincide and are satisfied because of the definitions of α and β.

As the basis of the proof of (5.62) we use the known result

$$\sup_{n\geq 2} \frac{\left\|L^{\alpha,\beta}_{n,r,s}\right\|_{\mathcal{L}(\mathbf{C}_{\tau_{N+1},\tau_0})}}{\ln n} < \infty \quad \text{if} \quad \begin{array}{l} \frac{\alpha}{2}+\frac{1}{4} \leq r+\tau_{N+1} \leq \frac{\alpha}{2}+\frac{5}{4}\,, \\ \frac{\beta}{2}+\frac{1}{4} \leq r+\tau_0 \leq \frac{\beta}{2}+\frac{5}{4} \end{array} \tag{5.63}$$

for the interpolation operators from Definition 5.10. The proof of (5.63) can be found in [CJLM, Section 4] for interpolation operators based on the zeros of Jacobi polynomials. Exactly the same proof can be written down if the zeros of orthogonal polynomials with respect to $v^{\alpha,\beta}h$, $0 < h \in \mathbf{C}^0$, are considered (see [L1]). We remark that, in view of (5.22), the assumptions

$$1 + x^{\alpha,\beta}_{n,0} \sim x^{\alpha,\beta}_{n,1} - x^{\alpha,\beta}_{n,0} \sim n^{-2} \quad \text{and} \quad 1 - x^{\alpha,\beta}_{n,n+1} \sim x^{\alpha,\beta}_{n,n+1} - x^{\alpha,\beta}_{n,n} \sim n^{-2} \tag{5.64}$$

given in [CJLM] are satisfied. First we mention that, for

$$r = r(\alpha,\beta,u) \quad \text{and} \quad s = s(\alpha,\beta,u)\,,$$

only the second inequalities have to be assumed in (5.63), i.e., under the assumptions of (5.62) we have

$$\left\|L^{\alpha,\beta}_{n,r,s}\right\|_{\mathcal{L}(\mathbf{C}_{\tau_{N+1},\tau_0})} \leq c\,\ln n\,, \quad n \geq 2\,. \tag{5.65}$$

For every $n \geq n_0$ (n_0 large enough such that (5.21) is satisfied for $n \geq n_0$) we define the following polynomial of degree N,

$$q_n(x) = \prod_{i=1}^{N}\left(x - x^{\alpha,\beta}_{n,m(i)}\right).$$

(If $N = 0$, then we set $q_n = 1$.) Then, for all $f \in \mathbf{C}_u$, fq_n vanishes in every $x^{\alpha,\beta}_{n,m(i)}$ (in the limit sense if $x^{\alpha,\beta}_{n,m(i)} = x_i$) and

$$L^{\alpha,\beta}_{n,u} f = q_n^{-1} L^{\alpha,\beta}_{n,r,s}(fq_n)\,, \quad n \geq n_0\,, \tag{5.66}$$

since the right hand side is a polynomial of degree $< n+r+s-N$ which coincides with f on $\left\{x^{\alpha,\beta}_{n,j}\right\}_{j=1-s}^{n+r} \setminus \left\{x^{\alpha,\beta}_{n,m(i)}\right\}_{i=1}^{N}$. Let $f_n = v^{-\tau_{N+1},-\tau_0} S_n$, where S_n denotes the linear spline which interpolates $v^{\tau_{N+1},\tau_0} f q_n$ in $x^{\alpha,\beta}_{n,j}$, $j = 1-s,\ldots,n+r$, and which vanishes in ± 1. Taking (5.66) and (5.65) into account, we obtain

$$\begin{aligned} \left\|L^{\alpha,\beta}_{n,u} f\right\|_{v^{\tau_{N+1},\tau_0} q_n} &= \left\|L^{\alpha,\beta}_{n,r,s} f_n\right\|_{\tau_{N+1},\tau_0} \\ &\leq c\,\|f_n\|_{\tau_{N+1},\tau_0} \ln n \;=\; c \max_{j=1-s,\ldots,n+r} \left|\left(v^{\tau_{N+1},\tau_0}\, f q_n\right)\left(x^{\alpha,\beta}_{n,j}\right)\right| \ln n\,. \end{aligned}$$

We have $\left|x^{\alpha,\beta}_{n,j} - x^{\alpha,\beta}_{n,m(i)}\right| \leq \left|x^{\alpha,\beta}_{n,j} - x_i\right| + \left|x_i - x^{\alpha,\beta}_{n,m(i)}\right| \leq 2\left|x^{\alpha,\beta}_{n,j} - x_i\right|$ for all i,j. Consequently, $\left|q_n\left(x^{\alpha,\beta}_{n,j}\right)\right| \leq 2^N \prod_{i=1}^{N} \left|x^{\alpha,\beta}_{n,j} - x_i\right| \leq c \prod_{i=1}^{N} \left|x^{\alpha,\beta}_{n,j} - x_i\right|^{\tau_i}$ and we conclude

$$\left\|L^{\alpha,\beta}_{n,u} f\right\|_{v^{\tau_{N+1},\tau_0} q_n} \leq c\,\|f\|_u \ln n\,.$$

It remains to prove $\left\|L_{n,u}^{\alpha,\beta}f\right\|_u \le c\,n^{\max\{1-\tau_1,\dots,1-\tau_N\}}\left\|L_{n,u}^{\alpha,\beta}f\right\|_{v^{\tau_{N+1},\tau_0}q_n}$. Let $i\in\{1,\dots,N\}$. In view of (5.22), the distance between the smallest $x_{n,j}^{\alpha,\beta}\ge x_i$ and the biggest $x_{n,j}^{\alpha,\beta}<x_i$ can be estimated by $c\,n^{-1}$. Thus, there exists a constant $C>0$ such that

$$\left|x_{n,m(i)}^{\alpha,\beta}-x_i\right| \le \frac{C}{2n} \quad \text{for all } \ i=1,\dots,N \ \text{ and all } \ n\in\mathbb{N}\,. \tag{5.67}$$

If n_0 is chosen large enough, then, with the same constant C,

$$\left\|L_{n,u}^{\alpha,\beta}f\right\|_u \sim \left\|u\,L_{n,u}^{\alpha,\beta}f\right\|_{\mathbf{C}\left([-1,1]\setminus\bigcup\limits_{i=1}^{N}(x_i-Cn^{-1},\,x_i+Cn^{-1})\right)}\,,\quad n\ge n_0$$

(see (1.69)). For $i=1,\dots,N$ and $x\notin\bigcup_{i=1}^N(x_i-Cn^{-1},\,x_i+Cn^{-1})$ we have, by (5.67),

$$|x-x_i| \le \left|x-x_{n,m(i)}^{\alpha,\beta}\right| + \left|x_{n,m(i)}^{\alpha,\beta}-x_i\right| \le \left|x-x_{n,m(i)}^{\alpha,\beta}\right| + \frac{|x-x_i|}{2}$$

and, consequently,

$$\begin{aligned} u(x) \ &= \ \prod_{i=1}^N |x-x_i|^{\tau_i-1}\,v^{\tau_{N+1},\tau_0}(x)\prod_{i=1}^N|x-x_i| \le c\,n^{\max_{i=1}^N(1-\tau_i)}\,v^{\tau_{N+1},\tau_0}(x)\prod_{i=1}^N|x-x_i| \\ &\le \ c\,n^{\max_{i=1}^N(1-\tau_i)}\,v^{\tau_{N+1},\tau_0}(x)\prod_{i=1}^N\left|x-x_{n,m(i)}^{\alpha,\beta}\right| = c\,n^{\max_{i=1}^N(1-\tau_i)}\left(v^{\tau_{N+1},\tau_0}q_n\right)(x). \end{aligned}$$

Thus, $\left\|L_{n,u}^{\alpha,\beta}f\right\|_u \le c\,n^{\max\{1-\tau_1,\dots,1-\tau_N\}}\left\|L_{n,u}^{\alpha,\beta}f\right\|_{v^{\tau_{N+1},\tau_0}q_n}$ and (5.62) is proved. ■

For the proof of (5.60) and (5.61) we need some auxiliary results.

Lemma 5.32 *Let $x_0\in(-1,1)$ and $\mu,\nu,\rho\in[0,1]$ be fixed. Then, for all $x\in(-1,1)$ and all $n\ge 2$,*

$$\sum_{\substack{i=1\\ i\ne k(x)}}^n \frac{\lambda_{n,i}^{\alpha,\beta}\,v^{-\alpha-\mu,-\beta-\nu}\left(x_{n,i}^{\alpha,\beta}\right)}{\left|x-x_{n,i}^{\alpha,\beta}\right|\left(\left|x_0-x_{n,i}^{\alpha,\beta}\right|+n^{-1}\right)^\rho} \le c\,\frac{\ln n}{v^{\mu,\nu}(x)\left(|x-x_0|+n^{-1}\right)^\rho}$$

$(c\ne c(n,x))$, where $k(x)\in\{0,\dots,n+1\}$ denotes the index of that $x_{n,k}^{\alpha,\beta}$, $k=0,\dots,n+1$, which is closest to x. (If there are two such $x_{n,k}^{\alpha,\beta}$, take the smaller one.)

Proof. It is well known (see [N, Theorem 9.22]) that $\theta_{n,i}-\theta_{n,i+1}\sim n^{-1}$, $i=0,\dots,n$, where $\theta_{n,i}:=\arccos x_{n,i}^{\alpha,\beta}$, $i=1,\dots,n$, and $\theta_{n,0}:=\pi$, $\theta_{n,n+1}:=0$. Consequently,

$$\sqrt{1-x_{n,i}^{\alpha,\beta}} = \sqrt{2}\sin\frac{\theta_{n,i}}{2} \sim \theta_{n,i} = \sum_{j=i}^n(\theta_{n,j}-\theta_{n,j+1}) \sim \frac{n+1-i}{n}\,,\quad i=1,\dots,n\,, \tag{5.68}$$

$$\sqrt{1+x_{n,i}^{\alpha,\beta}} = \sqrt{2}\sin\frac{\pi-\theta_{n,i}}{2} \sim \pi-\theta_{n,i} = \sum_{j=1}^i(\theta_{n,j-1}-\theta_{n,j}) \sim \frac{i}{n}\,,\quad i=1,\dots,n\,, \tag{5.69}$$

$$x_{n,i+1}^{\alpha,\beta}-x_{n,i}^{\alpha,\beta} = 2\sin\frac{\theta_{n,i}-\theta_{n,i+1}}{2}\sin\frac{\theta_{n,i}+\theta_{n,i+1}}{2} \sim \frac{\varphi\left(x_{n,i}^{\alpha,\beta}\right)}{n}\,,\quad i=1,\dots,n-1\,. \tag{5.70}$$

Together with (5.64) we obtain

$$x_{n,i}^{\alpha,\beta} - x_{n,i-1}^{\alpha,\beta} \sim \frac{\varphi\big(x_{n,i-1}^{\alpha,\beta}\big)}{n} \sim \frac{\varphi\big(x_{n,i}^{\alpha,\beta}\big)}{n} \sim x_{n,i+1}^{\alpha,\beta} - x_{n,i}^{\alpha,\beta}\,, \quad i = 1, \ldots, n\,. \tag{5.71}$$

Moreover, $\lambda_{n,i}^{\alpha,\beta} \le c\, n^{-1} v^{\alpha+(1/2),\beta+(1/2)}\big(x_{n,i}^{\alpha,\beta}\big)$ ([N, Theorem 6.3.28]). In view of (5.71), this can be rewritten as follows,

$$\lambda_{n,i}^{\alpha,\beta} \le c\, v^{\alpha,\beta}\big(x_{n,i}^{\alpha,\beta}\big) \min\big\{x_{n,i+1}^{\alpha,\beta} - x_{n,i}^{\alpha,\beta},\, x_{n,i}^{\alpha,\beta} - x_{n,i-1}^{\alpha,\beta}\big\}\,, \quad i = 1, \ldots, n\,. \tag{5.72}$$

Choose some constant $C_0 \in (0,1)$ such that $[x_0 - 3C_0,\, x_0 + 3C_0] \subset (-1,1)$. Let $x \in (-1,1)$. Then, by (5.72),

$$\begin{aligned}
&\sum_{\substack{i=1\\ i\ne k(x)}}^{n} \frac{\lambda_{n,i}^{\alpha,\beta}\, v^{-\alpha-\mu,-\beta-\nu}\big(x_{n,i}^{\alpha,\beta}\big)}{\big|x - x_{n,i}^{\alpha,\beta}\big|\,\big(\big|x_0 - x_{n,i}^{\alpha,\beta}\big| + n^{-1}\big)^{\rho}} \\
&\qquad \le\; c \sum_{\substack{i\in\{1,\ldots,n\}\setminus\{k(x)\}:\\ |x_{n,i}^{\alpha,\beta} - x_0| \le C_0|x - x_0|}} \frac{\big(x_{n,i+1}^{\alpha,\beta} - x_{n,i}^{\alpha,\beta}\big)\, v^{-\mu,-\nu}\big(x_{n,i}^{\alpha,\beta}\big)}{\big|x - x_{n,i}^{\alpha,\beta}\big|\,\big(\big|x_0 - x_{n,i}^{\alpha,\beta}\big| + n^{-1}\big)^{\rho}} \\
&\qquad\quad + c \sum_{\substack{i\in\{1,\ldots,n\}\setminus\{k(x)\}:\\ |x_{n,i}^{\alpha,\beta} - x_0| > C_0|x - x_0|,\ x_{n,i}^{\alpha,\beta} \le x}} \frac{\big(x_{n,i}^{\alpha,\beta} - x_{n,i-1}^{\alpha,\beta}\big)\, v^{-\mu,-\nu}\big(x_{n,i}^{\alpha,\beta}\big)}{\big|x - x_{n,i}^{\alpha,\beta}\big|\,\big(\big|x_0 - x_{n,i}^{\alpha,\beta}\big| + n^{-1}\big)^{\rho}} \\
&\qquad\quad + c \sum_{\substack{i\in\{1,\ldots,n\}\setminus\{k(x)\}:\\ |x_{n,i}^{\alpha,\beta} - x_0| > C_0|x - x_0|,\ x_{n,i}^{\alpha,\beta} > x}} \frac{\big(x_{n,i+1}^{\alpha,\beta} - x_{n,i}^{\alpha,\beta}\big)\, v^{-\mu,-\nu}\big(x_{n,i}^{\alpha,\beta}\big)}{\big|x - x_{n,i}^{\alpha,\beta}\big|\,\big(\big|x_0 - x_{n,i}^{\alpha,\beta}\big| + n^{-1}\big)^{\rho}} \\
&\qquad =:\; c\, S_1 + c\, S_2 + c\, S_3\,.
\end{aligned}$$

For $i \in \{1, \ldots, n\} \setminus \{k(x)\}$ with $\big|x_{n,i}^{\alpha,\beta} - x_0\big| \le C_0|x - x_0|$ we have

$$x_{n,i}^{\alpha,\beta} \in [x_0 - 2C_0,\, x_0 + 2C_0] \quad \text{and, consequently,} \quad v^{-\mu,-\nu}\big(x_{n,i}^{\alpha,\beta}\big) \sim 1\,.$$

If $x \notin [x_0 - 3C_0,\, x_0 + 3C_0]$, then, for the same values of i, $\big|x - x_{n,i}^{\alpha,\beta}\big| \ge C_0$. Otherwise, $\big|x - x_{n,i}^{\alpha,\beta}\big| \ge 2^{-1} \min\big\{x_{n,k(x)+1}^{\alpha,\beta} - x_{n,k(x)}^{\alpha,\beta},\, x_{n,k(x)}^{\alpha,\beta} - x_{n,k(x)-1}^{\alpha,\beta}\big\} \ge c\, n^{-1}$ in view of (5.71). Thus,

$$\big|x - x_{n,i}^{\alpha,\beta}\big| \ge \frac{|x - x_0| - \big|x_0 - x_{n,i}^{\alpha,\beta}\big|}{2} + \frac{c}{n} \ge \frac{1 - C_0}{2}\,|x - x_0| + \frac{c}{n} \ge c\,\big(|x - x_0| + n^{-1}\big)$$

for all $i \in \{1, \ldots, n\} \setminus \{k(x)\}$ with $\big|x_{n,i}^{\alpha,\beta} - x_0\big| \le C_0\,|x - x_0|$. Moreover, by (5.71),

$$\begin{aligned}
\int_{x_{n,i}^{\alpha,\beta}}^{x_{n,i+1}^{\alpha,\beta}} \frac{dt}{\big(|t - x_0| + n^{-1}\big)^{\rho}} &\ge \int_{x_{n,i}^{\alpha,\beta}}^{x_{n,i+1}^{\alpha,\beta}} \frac{dt}{\big(\big|t - x_{n,i}^{\alpha,\beta}\big| + \big|x_{n,i}^{\alpha,\beta} - x_0\big| + n^{-1}\big)^{\rho}} \\
&\ge c \int_{x_{n,i}^{\alpha,\beta}}^{x_{n,i+1}^{\alpha,\beta}} \frac{dt}{\big(\big|x_{n,i}^{\alpha,\beta} - x_0\big| + n^{-1}\big)^{\rho}} = c\, \frac{x_{n,i+1}^{\alpha,\beta} - x_{n,i}^{\alpha,\beta}}{\big(\big|x_{n,i}^{\alpha,\beta} - x_0\big| + n^{-1}\big)^{\rho}}\,.
\end{aligned} \tag{5.73}$$

Consequently,

$$S_1 \le \frac{c}{\left(|x-x_0|+n^{-1}\right)} \sum_{\substack{i\in\{1,\dots,n\}\setminus\{k(x)\}:\\ |x_{n,i}^{\alpha,\beta}-x_0|\le C_0|x-x_0|}} \int_{x_{n,i}^{\alpha,\beta}}^{x_{n,i+1}^{\alpha,\beta}} \frac{dt}{\left(|t-x_0|+n^{-1}\right)^{\rho}}\,.$$

The above integrals are taken over subintervals of $[x_0-C_0|x-x_0|,\ x_0+C_0|x-x_0|+Cn^{-1}]$, $C>0$ some constant, since $\left|x_{n,i}^{\alpha,\beta}-x_0\right|\le C_0|x-x_0|$ and $x_{n,i+1}^{\alpha,\beta}-x_{n,i}^{\alpha,\beta}\le Cn^{-1}$. Hence,

$$S_1 \le \frac{c}{|x-x_0|+n^{-1}} \int_{x_0-C_0|x-x_0|}^{x_0+C_0|x-x_0|+Cn^{-1}} \frac{dt}{\left(|t-x_0|+n^{-1}\right)^{\rho}}\,.$$

The last integral is bounded by $c\left(|x-x_0|+n^{-1}\right)^{1-\rho}$ if $\rho<1$ and by $c\ln n$ if $\rho=1$. Thus,

$$S_1 \le c\,\frac{\ln n}{\left(|x-x_0|+n^{-1}\right)^{\rho}}\,.$$

For $i\in\{1,\dots,n\}\setminus\{k(x)\}$ with $\left|x_{n,i}^{\alpha,\beta}-x_0\right|>C_0|x-x_0|$ and $x_{n,i}^{\alpha,\beta}\le x$ we have, by (5.71),

$$\frac{1}{\left(\left|x_0-x_{n,i}^{\alpha,\beta}\right|+n^{-1}\right)^{\rho}} \le \frac{c}{\left(|x-x_0|+n^{-1}\right)^{\rho}}\,,$$

$$\left(1-x_{n,i}^{\alpha,\beta}\right)^{-\mu}\le(1-x)^{-\mu}\,,\quad x-x_{n,i}^{\alpha,\beta}\ge\frac{x_{n,i+1}^{\alpha,\beta}-x_{n,i}^{\alpha,\beta}}{2}\sim x_{n,i}^{\alpha,\beta}-x_{n,i-1}^{\alpha,\beta}\ge\frac{c}{n^2}\,,$$

$$\left(1+x_{n,i}^{\alpha,\beta}\right)^{-\nu}\le(1+t)^{-\nu}\,,\ x-t\le\left(x-x_{n,i}^{\alpha,\beta}\right)+\left(x_{n,i}^{\alpha,\beta}-x_{n,i-1}^{\alpha,\beta}\right)\quad\text{for}\ \ t\in\left[x_{n,i-1}^{\alpha,\beta},x_{n,i}^{\alpha,\beta}\right].$$

In particular, $x-x_{n,i}^{\alpha,\beta}\ge c\,(x-t+n^{-2})$ for all $t\in\left[x_{n,i-1}^{\alpha,\beta},x_{n,i}^{\alpha,\beta}\right]$. Consequently,

$$S_2 \le c\,\frac{(1-x)^{-\mu}}{\left(|x-x_0|+n^{-1}\right)^{\rho}} \sum_{\substack{i\in\{1,\dots,n\}\setminus\{k(x)\}:\\ |x_{n,i}^{\alpha,\beta}-x_0|>C_0|x-x_0|,\ x_{n,i}^{\alpha,\beta}\le x}} \int_{x_{n,i-1}^{\alpha,\beta}}^{x_{n,i}^{\alpha,\beta}} \frac{(1+t)^{-\nu}}{x-t+n^{-2}}\,dt\,.$$

If $x<x_{n,1}^{\alpha,\beta}$, then $S_2=0$. If $x\ge x_{n,1}^{\alpha,\beta}$, then $(x-1)/2\ge-1+Cn^{-2}$ and $x_{n,0}^{\alpha,\beta}\ge-1+Cn^{-2}$ with some constant $C>0$ (see (5.69)). Thus, for $x\ge x_{n,1}^{\alpha,\beta}$,

$$\begin{aligned}
S_2 &\le c\,\frac{(1-x)^{-\mu}}{\left(|x-x_0|+n^{-1}\right)^{\rho}} \int_{-1+Cn^{-2}}^{x} \frac{(1+t)^{-\nu}}{x-t+n^{-2}}\,dt \\
&\le c\,\frac{(1-x)^{-\mu}}{\left(|x-x_0|+n^{-1}\right)^{\rho}} \left(2^{\nu}(1+x)^{-\nu}\int_{(x-1)/2}^{x}\frac{dt}{x-t+n^{-2}}\right. \\
&\qquad\qquad\left. +\ \frac{2}{x+1}\int_{-1+Cn^{-2}}^{(x-1)/2}(1+t)^{-\nu}\,dt\right) \\
&\le c\,\frac{(1-x)^{-\mu}}{\left(|x-x_0|+n^{-1}\right)^{\rho}} \left((1+x)^{-\nu}\ln n+\frac{1}{x+1}\int_{-1+Cn^{-2}}^{(x-1)/2}(1+t)^{-\nu}\,dt\right).
\end{aligned}$$

The last integral is bounded by $c\,(1+x)^{1-\nu}$ if $\nu<1$ and by $c\ln n$ if $\nu=1$. Analogously, we obtain

$$S_3 \le c\,\frac{(1+x)^{-\nu}}{\big(|x-x_0|+n^{-1}\big)^{\rho}}\int_x^{1-Cn^{-2}} \frac{(1-t)^{-\mu}}{t-x+n^{-2}}\,dt \le c\,\frac{v^{-\mu,-\nu}(x)}{\big(|x-x_0|+n^{-1}\big)^{\rho}}\ln n$$

if $x \le x_{n,n}^{\alpha,\beta}$. If $x > x_{n,n}^{\alpha,\beta}$, then $S_3=0$. ■

Lemma 5.33 *Let $x_0\in(-1,1)$ and $\mu,\nu,\rho\in[0,1)$ be fixed. Then,*

$$\sum_{i=1}^n \frac{\lambda_{n,i}^{\alpha,\beta}\, v^{-\alpha-\mu,-\beta-\nu}\big(x_{n,i}^{\alpha,\beta}\big)}{\big(\big|x_0-x_{n,i}^{\alpha,\beta}\big|+n^{-1}\big)^{\rho}} \le c \quad \textit{for all} \;\; n\in\mathbb{N}\,,$$

where $c\ne c(n)$.

Proof. The estimates (5.72) and (5.73) hold for all $i=1,\dots,n$. Consequently,

$$\sum_{i=1}^n \frac{\lambda_{n,i}^{\alpha,\beta}\, v^{-\alpha-\mu,-\beta-\nu}\big(x_{n,i}^{\alpha,\beta}\big)}{\big(\big|x_0-x_{n,i}^{\alpha,\beta}\big|+n^{-1}\big)^{\rho}} \le c\sum_{i=1}^n v^{-\mu,-\nu}\big(x_{n,i}^{\alpha,\beta}\big)\int_{x_{n,i}^{\alpha,\beta}}^{x_{n,i+1}^{\alpha,\beta}} \frac{dt}{\big(|t-x_0|+n^{-1}\big)^{\rho}}\,.$$

In view of (5.68), (5.69), and (5.64) we have $v^{-\mu,-\nu}\big(x_{n,i}^{\alpha,\beta}\big)\sim v^{-\mu,-\nu}(t)$ for all i and all $t\in\big[x_{n,i}^{\alpha,\beta}, x_{n,i+1}^{\alpha,\beta}\big]$. Moreover, $\big(|t-x_0|+n^{-1}\big)^{\rho}\ge |t-x_0|^{\rho}$. Thus,

$$\sum_{i=1}^n v^{-\mu,-\nu}\big(x_{n,i}^{\alpha,\beta}\big)\int_{x_{n,i}^{\alpha,\beta}}^{x_{n,i+1}^{\alpha,\beta}} \frac{dt}{\big(|t-x_0|+n^{-1}\big)^{\rho}} \le c\int_{-1}^1 \frac{v^{-\mu,-\nu}(t)}{|t-x_0|^{\rho}}\,dt$$

and the lemma is proved. ■

Corollary 5.34 *Let u be a weight of the form (2.19), where even $0\le\tau_j\le 1$ is allowed, and set*

$$u_n(x) := v^{\tau_{N+1},\tau_0}(x)\prod_{k=1}^N \left(|x-x_k|+\frac{1}{n}\right)^{\tau_k}.$$

Further, let $k(x)$ be defined as in Lemma 5.32. There is a constant $c\ne c(n,x)$ such that

$$\sum_{\substack{i=1\\ i\ne k(x)}}^n \frac{\lambda_{n,i}^{\alpha,\beta}\, v^{-\alpha,-\beta}\big(x_{n,i}^{\alpha,\beta}\big)}{\big|x-x_{n,i}^{\alpha,\beta}\big|\, u_n\big(x_{n,i}^{\alpha,\beta}\big)} \le c\,\frac{\ln n}{u_n(x)} \qquad \textit{for all}\;\; x\in(-1,1) \;\; \textit{and all} \;\; n\ge 2\,.$$

If $0\le\tau_j<1$ for all $j=0,\dots,N+1$, then also the estimate

$$\sum_{i=1}^n \frac{\lambda_{n,i}^{\alpha,\beta}\, v^{-\alpha,-\beta}\big(x_{n,i}^{\alpha,\beta}\big)}{u_n\big(x_{n,i}^{\alpha,\beta}\big)} \le c\,, \quad n\in\mathbb{N}$$

holds true, where $c\ne c(n)$.

Proof. If $N = 0$, then the assertions follow from Lemmas 5.32 and 5.33 applied to $\mu = \tau_{N+1}$, $\nu = \tau_0$, and $\rho = 0$. If $N > 0$, then we use the equivalence

$$u_n^{-1}(x) \sim v^{-\tau_{N+1},-\tau_0}(x) \sum_{k=1}^{N} \left(|x - x_k| + \frac{1}{n} \right)^{-\tau_k}$$

(write the right hand side as a fraction) to obtain the assertions with the help of Lemmas 5.32 and 5.33 (applied to $\mu = \tau_{N+1}$, $\nu = \tau_0$, and $\rho = \tau_k$, $k = 1, \ldots, N$). ∎

Lemma 5.35 *Let $\rho, \tau \geq -1$ and $C \in (0, 1)$ be fixed. Then,*

$$\left\| v^{\rho,\tau} \varphi \, p_n' \right\|_{\mathbf{C}[-1+Cn^{-2},\,1-Cn^{-2}]} \leq c\, n\, \|p_n\|_{\rho,\tau}\,, \quad p_n \in \Pi_n \cap \mathbf{C}_{\rho,\tau}\,, \quad n \in \mathbb{N}\,,$$

where $c \neq c(n, p_n)$.

Proof. $p_n \in \Pi_n \cap \mathbf{C}_{\rho,\tau}$ implies $p_n = p\, q_n$ with $q_n \in \Pi_n$ and $p = v^{m(\rho),m(\tau)}$, where $m(\rho) = 0$ if $\rho \geq 0$ and $m(\rho) = 1$ if $\rho < 0$. Consequently,

$$\begin{aligned} &\left\| v^{\rho,\tau} \varphi \, p_n' \right\|_{\mathbf{C}[-1+Cn^{-2},\,1-Cn^{-2}]} \\ &\quad \leq c \left\| v^{\rho,\tau} \varphi \, q_n \right\|_{\mathbf{C}[-1+Cn^{-2},\,1-Cn^{-2}]} + \left\| v^{\rho+m(\rho),\tau+m(\tau)} \varphi \, q_n' \right\| . \end{aligned}$$

From the Bernstein inequality for the weight $v^{\rho+m(\rho),\tau+m(\tau)}$ (see Proposition 1.14) we conclude

$$\left\| v^{\rho+m(\rho),\tau+m(\tau)} \varphi \, q_n' \right\| \leq c\, n \left\| v^{\rho+m(\rho),\tau+m(\tau)}\, q_n \right\| = c\, n\, \|p_n\|_{\rho,\tau}\,.$$

Moreover, it is clear that $|\varphi\, q_n| \leq c\, n\, |p\, q_n| = c\, n\, |p_n|$ on $[-1 + Cn^{-2},\, 1 - Cn^{-2}]$. ∎

Lemma 5.36 *Let $\gamma > 1$, $\delta \leq 0$, $C \in (0, 1)$, and $\rho, \tau \geq -1$ be fixed. Then, for $f \in \mathbf{C}_{\rho,\tau}^{\gamma,\delta}$ and $p_n \in \Pi_n \cap \mathbf{C}_{\rho,\tau}$ with $\|f - p_n\|_{\rho,\tau} \leq M(f)\, n^{-\gamma} \ln^{-\delta} n$ ($n \geq 2$, $M \neq M(n)$),*

$$\left\| v^{\rho,\tau} \varphi \,(f' - p_n') \right\|_{\mathbf{C}[-1+Cn^{-2},\,1-Cn^{-2}]} \leq c\, \frac{M(f)}{n^{\gamma-1} \ln^{\delta} n}\,, \quad n \geq 2\,, \tag{5.74}$$

where $c \neq c(n, f, p_n)$. (Remark that, by assertion (ii) of Proposition 1.19, $f \in \mathbf{C}^1(-1, 1)$.)

Proof. For every $n \geq 2$ we have

$$f - p_n = \sum_{i=0}^{\infty} \left(p_{2^{i+1}n} - p_{2^i n} \right) \ \text{ in the norm of } \ \mathbf{C}[-1 + Cn^{-2},\, 1 - Cn^{-2}]\,. \tag{5.75}$$

(Use that $\|f - p_{2^j n}\|_{\mathbf{C}[-1+Cn^{-2},1-Cn^{-2}]} \leq c_n \|f - p_{2^j n}\|_{\rho,\tau}$.) The series (5.75) is even convergent in $\mathbf{C}^1[-1 + Cn^{-2},\, 1 - Cn^{-2}]$, since, by Lemma 5.35,

$$\begin{aligned} \sum_{i=0}^{\infty} \left\| v^{\rho,\tau} \varphi \,(p_{2^{i+1}n}' - p_{2^i n}') \right\|_{\mathbf{C}[-1+Cn^{-2},\,1-Cn^{-2}]} &\leq c\, n \sum_{i=0}^{\infty} \left\| p_{2^{i+1}n} - p_{2^i n} \right\|_{\rho,\tau} \\ &\leq c\, M(f)\, \frac{\ln^{-\delta} n}{n^{\gamma-1}} \sum_{i=0}^{\infty} \frac{(i+1)^{-\delta}}{2^{\gamma i}}\,. \end{aligned}$$

This estimate and (5.75) show that (5.74) holds true. ∎

Proof of (5.60). Let $f_n \in \Pi_{n+r+s-N}$, $x \in \left\{x^{-\alpha,-\beta}_{n(\alpha,\beta),j}\right\}_{j=1-s}^{n(\alpha,\beta)+r} \setminus \left\{x^{-\alpha,-\beta}_{n(\alpha,\beta),k(i)}\right\}_{i=1}^{N}$, and denote by $I_{\alpha,\beta}$ the functional $f \to I_{\alpha,\beta}(f) = \pi^{-1}\int_{-1}^{1} f(t)\,\sigma_{\alpha,\beta}(t)\,dt$. We have

$$\begin{aligned}\big[(\mathcal{K}-\mathcal{K}_n)f_n\big](x) &= I_{\alpha,\beta}\big(k(x,\,.\,)f_n\big) - Q_n^{\alpha,\beta}\big(k(x,\,.\,)f_n\big)\\ &= I_{\alpha,\beta}\big(k_1(x,\,.\,)f_n\big) - Q_n^{\alpha,\beta}\big(k_1(x,\,.\,)f_n\big) + I_{\alpha,\beta}\big(k_2(x,\,.\,)f_n\big) - Q_n^{\alpha,\beta}\big(k_2(x,\,.\,)f_n\big)\,.\end{aligned}$$

Let $k_{n1}(x,\,.\,) \in \Pi_{n-1} \cap \mathbf{C}_w$ such that $E^w_{n-1}(k_1(x,\,.\,)) = \|k_1(x,\,.\,) - k_{n1}(x,\,.\,)\|_w$. If we take into account that, by Proposition 5.1, (5.40) holds with sup instead of ess sup and that

$$\|f_n\|_{u(\alpha,\beta)} \sim \|v^{\alpha,\beta}u_n f_n\| \quad (u_n \text{ from Corollary 5.34}) \tag{5.76}$$

(see (1.69)), then we obtain, in view of (5.18),

$$\begin{aligned}&\Big|I_{\alpha,\beta}\big(k_1(x,\,.\,)f_n\big) - Q_n^{\alpha,\beta}\big(k_1(x,\,.\,)f_n\big)\Big|\\ &= \Big|I_{\alpha,\beta}\big([k_1(x,\,.\,)-k_{n1}(x,\,.\,)]f_n\big) - Q_n^{\alpha,\beta}\big([k_1(x,\,.\,)-k_{n1}(x,\,.\,)]f_n\big)\Big|\\ &\le \|k_1(x,\,.\,)-k_{n1}(x,\,.\,)\|_w\,\|f_n\|_{u(\alpha,\beta)}\Big(I_{\alpha,\beta}\big(w^{-1}u(\alpha,\beta)^{-1}\big) + Q_n^{\alpha,\beta}\big(w^{-1}v^{-\alpha,-\beta}u_n^{-1}\big)\Big)\\ &\le c\,u^{-1}(x)\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^\gamma \ln^{\delta+1} n}\left(1 + \sum_{i=1}^{n}\frac{\lambda^{\alpha,\beta}_{n,i}\,v^{-\alpha,-\beta}\big(x^{\alpha,\beta}_{n,i}\big)}{(v^{\mu,\nu}u_n)\big(x^{\alpha,\beta}_{n,i}\big)}\right).\end{aligned} \tag{5.77}$$

Clearly, $v^{\mu,\nu} \ge c\,v^{\max\{0,\mu\},\max\{0,\nu\}}$. Thus, Corollary 5.34, applied to $v^{\max\{0,\mu\},\max\{0,\nu\}}u$ instead of u, shows that the sum in (5.77) is uniformly bounded. Consequently,

$$\big|(\mathcal{K}-\mathcal{K}_n)f_n\big|(x) \le c\,u^{-1}(x)\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^\gamma \ln^{\delta+1} n} + \Big|I_{\alpha,\beta}\big(k_2(x,\,.\,)f_n\big) - Q_n^{\alpha,\beta}\big(k_2(x,\,.\,)f_n\big)\Big|\,.$$

To estimate the last term, we proceed similarly to the derivation of (5.77), where we take

$$k_{n2}(x,t) = \frac{p_n(x,t)-p_n(x,x)}{t-x} \quad \text{with} \quad p_n(x,\,.\,) \in \Pi_n\colon\ E_n(h_2(x,\,.\,)) = \|h_2(x,\,.\,) - p_n(x,\,.\,)\|\,.$$

The only difference to the above consideration is that now we decompose the integral $I_{\alpha,\beta}\big([k_2(x,\,.\,)-k_{n2}(x,\,.\,)]f_n\big)$ into two integrals over

$$I = \left[x - \frac{1+x}{n^\theta},\, x + \frac{1-x}{n^\theta}\right] \quad \text{and} \quad J = (-1,1)\setminus I\,, \tag{5.78}$$

where $\theta > 0$ is some sufficiently large constant. (Later we will see how big θ must be.) If we take into account that $k_{n2}(x,\,.\,) \in \Pi_{n-1}$ and $\big|h_2(x,t) - [p_n(x,t) - p_n(x,x)]\big| \le |h_2(x,t) - p_n(x,t)| + |p_n(x,x) - h_2(x,x)| \le c\,n^{-\gamma}\ln^{-\delta-2} n$ (where we used (5.40) with sup instead of ess sup; see Proposition 5.1), then we obtain

$$\begin{aligned}
&\Big| I_{\alpha,\beta}\big(k_2(x,\,.\,)f_n\big) - Q_n^{\alpha,\beta}\big(k_2(x,\,.\,)f_n\big)\Big| \\
&\qquad \le c\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^\gamma \ln^{\delta+2} n}\left(\int_J \frac{u^{-1}(t)}{|t-x|}\,dt + \sum_{i=1}^n \frac{\lambda_{n,i}^{\alpha,\beta}\, v^{-\alpha,-\beta}\big(x_{n,i}^{\alpha,\beta}\big)}{\big|x - x_{n,i}^{\alpha,\beta}\big|\, u_n\big(x_{n,i}^{\alpha,\beta}\big)}\right) \\
&\qquad\quad + c\,\|f_n\|_{u(\alpha,\beta)}\left(\int_I |k_2(x,t)|\,\frac{dt}{u(t)} + \int_I |k_{n2}(x,t)|\,\frac{dt}{u(t)}\right).
\end{aligned}$$

From Lemma 3.10, applied to the characteristic function g of J, it follows

$$\int_J \frac{u^{-1}(t)}{|t-x|}\,dt \le c\,u^{-1}(x)\left(1 + \int_J \frac{dt}{|t-x|}\right) \le c\,u^{-1}(x)\ln n\,.$$

Moreover, in view of Corollary 5.34,

$$\sum_{i=1}^n \frac{\lambda_{n,i}^{\alpha,\beta}\, v^{-\alpha,-\beta}\big(x_{n,i}^{\alpha,\beta}\big)}{\big|x - x_{n,i}^{\alpha,\beta}\big|\, u_n\big(x_{n,i}^{\alpha,\beta}\big)} \le \frac{\lambda_{n,k(x)}^{\alpha,\beta}\, v^{-\alpha,-\beta}\big(x_{n,k(x)}^{\alpha,\beta}\big)}{\big|x - x_{n,k(x)}^{\alpha,\beta}\big|\, u_n\big(x_{n,k(x)}^{\alpha,\beta}\big)} + c\,u^{-1}(x)\ln n\,,$$

where the first addend on the right hand side can be omitted if $k(x) \in \{0, n+1\}$. In particular, this is the case if $x = x_{n(\alpha,\beta),j}^{-\alpha,-\beta}$ with $j \in \{0, n(\alpha,\beta)+1\}$. If $x = x_{n(\alpha,\beta),j}^{-\alpha,-\beta}$ with $j \in \{1,\dots,n(\alpha,\beta)\}$ and $k(x) \in \{1,\dots,n\}$, then, by Proposition 5.12,

$$\frac{\lambda_{n,k(x)}^{\alpha,\beta}\, v^{-\alpha,-\beta}\big(x_{n,k(x)}^{\alpha,\beta}\big)}{\big|x - x_{n,k(x)}^{\alpha,\beta}\big|\, u_n\big(x_{n,k(x)}^{\alpha,\beta}\big)} \le c\,u_n^{-1}\big(x_{n,k(x)}^{\alpha,\beta}\big) \le c\,u^{-1}(x)\,,$$

where we took into account that, in view of (5.68)–(5.70), $1 \pm x \sim 1 \pm x_{n,k(x)}^{\alpha,\beta}$ and

$$|x - x_i| \le \big|x - x_{n,k(x)}^{\alpha,\beta}\big| + \big|x_{n,k(x)}^{\alpha,\beta} - x_i\big| \le c\left(n^{-1} + \big|x_{n,k(x)}^{\alpha,\beta} - x_i\big|\right), \quad i = 1,\dots,N\,.$$

Consequently,

$$\big|(\mathcal{K} - \mathcal{K}_n)f_n\big|(x) \le c\,\|f_n\|_{u(\alpha,\beta)}\left(\frac{u^{-1}(x)}{n^\gamma \ln^{\delta+1} n} + \int_I |k_2(x,t)|\,\frac{dt}{u(t)} + \int_I |k_{n2}(x,t)|\,\frac{dt}{u(t)}\right).$$

The last two integrals can be estimated similarly to the integrals I_1 and I_2 from the proof of Theorem 3.5; see (3.23) and (3.24). (Set $P = w = v = 1$, $h(x,t) = h_2(x,t)$, and $h_n(x,t) = p_n(x,t) - p_n(x,x)$ in I_1, I_2.) In this way we obtain

$$\int_I |k_2(x,t)|\,\frac{dt}{u(t)} + \int_I |k_{n2}(x,t)|\,\frac{dt}{u(t)} \le c\,\frac{u^{-1}(x)}{n^\gamma \ln^{\delta+1} n}$$

supposed that $\theta > \max\{\gamma+2,\, \gamma/\eta\}$, where η is a Hölder exponent of $h_2(x,t)$ (see Proposition 5.1).

$\blacksquare$

Proof of (5.61). Let $a_n, b_n \in \Pi_n$ such that $\|b - b_n\| = E_n(b)$ and

$$\left\|\widetilde{a}\,\sigma_{-\alpha,-\beta} - a_n\right\|_\psi = E_n^\psi\left(\widetilde{a}\,\sigma_{-\alpha,-\beta}\right)\,, \quad \psi = \begin{cases} v^{\alpha,\beta}\,, & b(1) \neq 0\,,\ b(-1) \neq 0\,, \\ v^{\alpha+(\gamma/4),\beta}\,, & b(1) = 0\,,\ b(-1) \neq 0\,, \\ v^{\alpha,\beta+(\gamma/4)}\,, & b(1) \neq 0\,,\ b(-1) = 0\,, \\ v^{\alpha+(\gamma/4),\beta+(\gamma/4)}\,, & b(1) = 0\,,\ b(-1) = 0\,. \end{cases}$$

(Remark that, by Lemma 5.13, $\widetilde{a}\,\sigma_{-\alpha,-\beta} \in \mathbf{C}_\psi$.) Further, define

$$h_n(x,t) = \frac{1}{r^2(x)\cos^2\left(\ell_{\alpha,\beta}(x)\right)}\left[\frac{a_n(x)\,b_n(t) - a_n(t)\,b_n(x)}{t - x}\right]$$

and

$$Q_{n,x}^{\alpha,\beta} f = \sum_{\substack{i=1 \\ i \neq d(x)}}^{n} \lambda_{n,i}^{\alpha,\beta}\, f\left(x_{n,i}^{\alpha,\beta}\right)\,, \tag{5.79}$$

where $d(x)$ is the index which appears in the definition (5.27) of $\mathcal{H}_n f$. More precisely,

$$d(x) = \begin{cases} k(x) & \text{if } \ a \notin \bigcup\limits_{\eta>1} \mathbf{H}^\eta([-1,1]) \ \text{ or } \ b \notin \bigcup\limits_{\eta>1} \mathbf{H}^\eta([-1,1]) \ \text{ or } \ b(-1)b(1) = 0\,, \\ 0 & \text{if } \ \gamma > 1 \ \text{ and } \ b(-1)b(1) \neq 0\,, \\ 0 \ \text{ or } \ k(x) & \text{if } \ \gamma < 1 \ \text{ and } \ b(-1)b(1) \neq 0 \ \text{ and } \ a, b \in \bigcup\limits_{\eta>1} \mathbf{H}^\eta([-1,1])\,, \end{cases}$$

where $k(x)$ is taken from Lemma 5.32. (Remark that, in view of Proposition 1.19, $\gamma > 1$ and $b(-1)b(1) \neq 0$ imply $a, b \in \bigcup_{\eta>1} \mathbf{H}^\eta([-1,1])$ because of assumption (5.33).) Let $f_n \in \Pi_{n+r+s-N}$, $x \in \left\{x_{n,j}^{\alpha,\beta}\right\}_{j=1-s}^{n+r} \setminus \left\{x_{n,m(i)}^{\alpha,\beta}\right\}_{i=1}^{N}$, and take the notation $I_{\alpha,\beta}$ from the proof of (5.60). Then, by (5.18),

$$\begin{aligned} &\left[(\mathcal{H} - \mathcal{H}_n) f_n\right](x) \\ &= I_{\alpha,\beta}\big([h(x,\,.\,) - h_n(x,\,.\,)]f_n\big) + (Q_n^{\alpha,\beta} - Q_{n,x}^{\alpha,\beta})\big(h_n(x,\,.\,)f_n\big) \\ &\quad + Q_{n,x}^{\alpha,\beta}\big([h_n(x,\,.\,) - h(x,\,.\,)]f_n\big) + \widetilde{b}(x) I_{\alpha,\beta}\big([b - b_{n-1}]f_n\big) + \widetilde{b}(x) Q_n^{\alpha,\beta}\big([b_{n-1} - b]f_n\big)\,. \end{aligned} \tag{5.80}$$

In view of assertion (i) of Corollary 1.18, we have $b \in \mathbf{C}^{\gamma,\widetilde{\delta}+1}$, i.e.,

$$\|b - b_n\| \leq \frac{c}{n^\gamma \ln^{\widetilde{\delta}+1} n}\,. \tag{5.81}$$

Together with (5.76) and Corollary 5.34 we obtain

$$\begin{aligned} &\left|\widetilde{b}(x) I_{\alpha,\beta}\big([b - b_{n-1}]f_n\big) + \widetilde{b}(x) Q_n^{\alpha,\beta}\big([b_{n-1} - b]f_n\big)\right| \\ &\qquad \leq c\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^\gamma \ln^{\widetilde{\delta}+1} n}\left(I_{\alpha,\beta}\big(u(\alpha,\beta)^{-1}\big) + Q_n^{\alpha,\beta}\big(v^{-\alpha,-\beta} u_n^{-1}\big)\right) \leq c\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^\gamma \ln^{\widetilde{\delta}+1} n}\,. \end{aligned} \tag{5.82}$$

By Lemma 2.28 and Lemma 5.13, $\widetilde{a}\,\sigma_{-\alpha,-\beta} = r^2\cos^2(\ell_{\alpha,\beta}) \cdot r^{-2}\sigma_{\alpha,\beta}^{-1}\,\widetilde{a} \in \mathbf{C}_\psi^{\widetilde{\gamma},\widetilde{\delta}+1}$, i.e.,

$$\left\|\widetilde{a}\,\sigma_{-\alpha,-\beta} - a_n\right\|_\psi \leq \frac{c}{n^{\widetilde{\gamma}} \ln^{\widetilde{\delta}+1} n}\,. \tag{5.83}$$

From (5.81) and (5.83) it follows

$$\big|(t-x)\big[h(x,t)-h_n(x,t)\big]\big| \le c\big(|b(t)|\,|(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x) - a_n(x)| + |a_n(x)|\,|b(t)-b_n(t)| + |b_n(x)|\,|a_n(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)| + |(\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)|\,|b_n(x)-b(x)|\big)$$

$$\le \frac{c}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}+1} n}\left(\frac{|b(t)|}{\psi(x)} + \frac{|b_n(x)|}{\psi(t)}\right) + \frac{c}{n^{\gamma}\ln^{\widetilde{\delta}+1} n}\Big(|a_n(x)| + v^{-\alpha,-\beta}(t)\Big).$$

We have $b \in \mathbf{C}^{\gamma/2,0}$ and, consequently, $b \in \mathbf{H}^{\gamma/4}([-1,1])$ (see assertion (iv) of Proposition 1.19). Thus, $|b(t)| \le c\,(1 \mp t)^{\gamma/4}$ if $b(\pm 1) = 0$, i.e.,

$$|b(t)| \le c\,\psi(t)\,v^{-\alpha,-\beta}(t) \quad \text{for all} \ \ t \in (-1,1)\,.$$

By (5.81), (5.68), and (5.69), $|b_n(x) - b(x)| \le c\,n^{-\gamma/2} \le c\,(1 \pm x)^{\gamma/4}$ and we conclude

$$|b_n(x)| \le c\,\psi(x)\,v^{-\alpha,-\beta}(x)\,. \tag{5.84}$$

From (5.83) it follows, using again $n^{-\gamma/2} \le c\,\psi(x)\,v^{-\alpha,-\beta}(x)$,

$$\frac{|a_n(x)|}{n^{\gamma-\widetilde{\gamma}}} \le \frac{|a_n(x) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)| + c\,v^{-\alpha,-\beta}(x)}{n^{\gamma-\widetilde{\gamma}}} \le c\,\frac{\psi^{-1}(x)}{n^{\gamma/2}} + c\,v^{-\alpha,-\beta}(x) \le c\,v^{-\alpha,-\beta}(x).$$

Together with (5.76) we obtain

$$\begin{aligned} &\big|\big[h(x,t) - h_n(x,t)\big]f_n(t)\big| \\ &\le \frac{c}{|t-x|}\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}+1} n}\left(\frac{1}{\psi(x)}\,\frac{\psi(t)}{(v^{2\alpha,2\beta}u_n)(t)} + \frac{\psi(x)}{v^{\alpha,\beta}(x)}\,\frac{1}{(\psi\,v^{\alpha,\beta}u_n)(t)}\right. \\ &\qquad \left. + \frac{1}{v^{\alpha,\beta}(x)}\,\frac{1}{(v^{\alpha,\beta}u_n)(t)} + \frac{1}{(v^{2\alpha,2\beta}u_n)(t)}\right). \end{aligned} \tag{5.85}$$

Define $\widetilde{Q}^{\alpha,\beta}_{n,x}$ by replacing $d(x)$ by $k(x)$ in (5.79). Then, by (5.85) and Corollary 5.34,

$$\begin{aligned} &\big|\widetilde{Q}^{\alpha,\beta}_{n,x}\big([h_n(x,\,.\,) - h(x,\,.\,)]f_n\big)\big| \\ &\le c\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}+1} n}\sum_{\substack{i=1\\ i\ne k(x)}}^{n}\frac{\lambda^{\alpha,\beta}_{n,i}\,v^{-\alpha,-\beta}\big(x^{\alpha,\beta}_{n,i}\big)}{\big|x - x^{\alpha,\beta}_{n,i}\big|}\left(\frac{1}{\psi(x)}\,\frac{\psi\big(x^{\alpha,\beta}_{n,i}\big)}{(v^{\alpha,\beta}u_n)\big(x^{\alpha,\beta}_{n,i}\big)}\right. \\ &\qquad \left. + \frac{\psi(x)}{v^{\alpha,\beta}(x)}\,\frac{1}{(\psi\,u_n)\big(x^{\alpha,\beta}_{n,i}\big)} + \frac{1}{v^{\alpha,\beta}(x)}\,\frac{1}{u_n\big(x^{\alpha,\beta}_{n,i}\big)} + \frac{1}{(v^{\alpha,\beta}u_n)\big(x^{\alpha,\beta}_{n,i}\big)}\right) \\ &\le \frac{c}{(v^{\alpha,\beta}u)(x)}\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}} n}\,. \end{aligned} \tag{5.86}$$

We have $x = x^{\alpha,\beta}_{n,j}$ ($j \in \{1-s,\dots,n+r\} \setminus \{m(1),\dots,m(N)\}$) and, consequently, $k(x) = j$. Thus, $\widetilde{Q}^{\alpha,\beta}_{n,x} \ne Q^{\alpha,\beta}_{n,x}$ is only possible if $j \in \{1,\dots,n\}$, $a', b' \in \mathbf{C}^{\eta,\min\{\delta+1,0\}}$ with some $\eta \in (0,1)$, and $b(-1)b(1) \ne 0$. In this case we estimate

$$\big|\big(Q^{\alpha,\beta}_{n,x} - \widetilde{Q}^{\alpha,\beta}_{n,x}\big)\big([h_n(x,\,.\,) - h(x,\,.\,)]f_n\big)\big| \le c\,\|f_n\|_{u(\alpha,\beta)}\,\frac{\lambda^{\alpha,\beta}_{n,j}}{(v^{\alpha,\beta}u)(x)}\,|h_n(x,x) - h(x,x)|\,.$$

From the formula (5.28) for $h(x,x)$ and the corresponding formula for $h_n(x,x)$ it follows

$$\begin{aligned} |h_n(x,x)-h(x,x)| &\le c\left|b_n' a_n - b_n a_n' - b'\,\widetilde{a}\,\sigma_{-\alpha,-\beta} + b\,(\widetilde{a}\,\sigma_{-\alpha,-\beta})'\right|(x) \\ &\le c\Big(\left|b_n'\,(a_n-\widetilde{a}\,\sigma_{-\alpha,-\beta})\right|(x) + \left|(b_n'-b')\,\widetilde{a}\,\sigma_{-\alpha,-\beta})\right|(x) \\ &\qquad + \left|b\,\left((\widetilde{a}\,\sigma_{-\alpha,-\beta})' - a_n'\right)\right|(x) + \left|(b-b_n)\,a_n'\right|(x)\Big)\,. \end{aligned}$$

If we take into account that (5.83) and (5.81) hold true with γ and $\widetilde{\delta}$ replaced by $\eta+1$ and $\min\{\delta,-1\}$, then we obtain, in view of Lemma 5.36 (which can be applied, since $x\in[-1+Cn^{-2},1-Cn^{-2}]$ because of (5.64)),

$$|h_n(x,x)-h(x,x)| \le \frac{c}{n^{\eta}\ln^{\min\{\delta+1,0\}} n}\left(\frac{v^{-\alpha,-\beta}(x)\,|b_n'(x)|}{n} + \frac{v^{-\alpha,-\beta}(x)}{\varphi(x)} + \frac{|a_n'(x)|}{n}\right).$$

Together with $\lambda_{n,j}^{\alpha,\beta} \le c\,n^{-1}v^{\alpha+(1/2),\beta+(1/2)}(x)$ ((5.72) and (5.71)) and Lemma 5.35 we get

$$\left|\left(Q_{n,x}^{\alpha,\beta}-\widetilde{Q}_{n,x}^{\alpha,\beta}\right)\left([h_n(x,\,.\,)-h(x,\,.\,)]f_n\right)\right| \le \frac{c}{(v^{\alpha,\beta}u)(x)}\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^{\eta+1}\ln^{\min\{\delta+1,0\}} n}\,. \tag{5.87}$$

If $\gamma>1$ and $\widetilde{\delta}=\min\{\delta,-1\}$, then $\eta=\gamma-1$. If $\gamma>1$ and $\widetilde{\delta}\ne\min\{\delta,-1\}$, then $\eta=\gamma-1+\varepsilon$. Thus, in every case (also if $\gamma<1$), (5.80), (5.82), (5.86), and (5.87) yield

$$\begin{aligned} &\left|(\mathcal{H}-\mathcal{H}_n)f_n\right|(x) \\ &\le \left|I_{\alpha,\beta}\left([h(x,\,.\,)-h_n(x,\,.\,)]f_n\right)\right| + \left|(Q_n^{\alpha,\beta}-Q_{n,x}^{\alpha,\beta})\left(h_n(x,\,.\,)f_n\right)\right| + \frac{c}{(v^{\alpha,\beta}u)(x)}\,\frac{\|f_n\|_{u(\alpha,\beta)}}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}} n}\,. \end{aligned}$$

If $\widetilde{\gamma}>1$, then $Q_n^{\alpha,\beta}=Q_{n,x}^{\alpha,\beta}$. If $\widetilde{\gamma}<1$, then we use again $\lambda_{n,j}^{\alpha,\beta}\le c\,n^{-1}v^{\alpha+(1/2),\beta+(1/2)}(x)$ to obtain

$$\begin{aligned} \left|\left(Q_n^{\alpha,\beta}-Q_{n,x}^{\alpha,\beta}\right)\left(h_n(x,\,.\,)f_n\right)\right| &\le c\,\|f_n\|_{u(\alpha,\beta)}\,\frac{\lambda_{n,j}^{\alpha,\beta}}{(v^{\alpha,\beta}u)(x)}\,|h_n(x,x)| \\ &\le c\,\frac{\|f_n\|_{u(\alpha,\beta)}}{u(x)}\,\frac{\varphi(x)}{n}\left(|b_n'a_n|(x)+|b_na_n'|(x)\right). \end{aligned}$$

Define $k\in\mathbb{N}$ by $2^k\le n<2^{k+1}$. If we write $a_n'=a_n'-a_{2^{k+1}}'+\sum_{l=0}^k\left(a_{2^{l+1}}'-a_{2^l}'\right)$, then we see that Lemma 5.35 and (5.83) imply

$$\begin{aligned} |a_n'(x)| &\le \frac{c}{\varphi(x)\psi(x)}\left(n\left\|a_n-a_{2^{k+1}}\right\|_\psi + \sum_{l=0}^k 2^l\left\|a_{2^{l+1}}-a_{2^l}\right\|_\psi\right) \\ &\le \frac{c}{\varphi(x)\psi(x)}\sum_{l=0}^k 2^{l(1-\widetilde{\gamma})}(l+1)^{-\widetilde{\delta}-1} \le c\,\frac{n^{1-\widetilde{\gamma}}\ln^{-\widetilde{\delta}-1} n}{\varphi(x)\psi(x)}\,. \end{aligned}$$

(Here we used that $\left\{2^{l\varepsilon}(l+1)^{-\widetilde{\delta}-1}\right\}$ is increasing for $l\ge l_0$.) Together with (5.84) we obtain

$$|b_na_n'|(x)\le c\,\frac{n^{1-\widetilde{\gamma}}\ln^{-\widetilde{\delta}-1} n}{\varphi(x)\,v^{\alpha,\beta}(x)}\,.$$

Analogously one can show that (5.81) and $n^{\widetilde{\gamma}-\gamma}|a_n(x)| \le c\, v^{-\alpha,-\beta}(x)$ (see the consideration after (5.84)) imply

$$|b_n' a_n|(x) \le c\, \frac{n^{1-\widetilde{\gamma}} \ln^{-\widetilde{\delta}-1} n}{\varphi(x)\, v^{\alpha,\beta}(x)} \,.$$

Thus, $\left|\left(Q_n^{\alpha,\beta} - Q_{n,x}^{\alpha,\beta}\right)\left(h_n(x,\,.\,) f_n\right)\right| \le c\, (v^{\alpha,\beta}u)^{-1}(x)\, \|f_n\|_{u(\alpha,\beta)}\, n^{-\widetilde{\gamma}} \ln^{-\widetilde{\delta}-1} n$ and we conclude

$$\left[(\mathcal{H} - \mathcal{H}_n) f_n\right](x) \le I_{\alpha,\beta}\left([h(x,\,.\,) - h_n(x,\,.\,)] f_n\right) + \frac{c}{(v^{\alpha,\beta}u)(x)}\, \frac{\|f_n\|_{u(\alpha,\beta)}}{n^{\widetilde{\gamma}} \ln^{\widetilde{\delta}} n} \,.$$

Let I and J be defined by (5.78) (with sufficiently large θ). Then, by (5.85),

$$\begin{aligned}
&\left|I_{\alpha,\beta}\left([h(x,\,.\,) - h_n(x,\,.\,)] f_n\right|\right. \\
&\quad \le c\, \frac{\|f_n\|_{u(\alpha,\beta)}}{n^{\widetilde{\gamma}} \ln^{\widetilde{\delta}+1} n} \left(\frac{1}{\psi(x)} \int_J \frac{\psi(t)}{|t-x|}\, \frac{dt}{(v^{\alpha,\beta}u)(t)} + \frac{\psi(x)}{v^{\alpha,\beta}(x)} \int_J \frac{1}{|t-x|}\, \frac{dt}{(\psi u)(t)} \right. \\
&\qquad\qquad \left. + \frac{1}{v^{\alpha,\beta}(x)} \int_J \frac{1}{|t-x|}\, \frac{dt}{u(t)} + \int_J \frac{1}{|t-x|}\, \frac{dt}{(v^{\alpha,\beta}u)(t)} \right) \\
&\qquad + c\, \|f_n\|_{u(\alpha,\beta)} \int_I \left| \frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\, b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\, b(x)}{t-x} \right| \frac{dt}{u(t)} \\
&\qquad + c\, \|f_n\|_{u(\alpha,\beta)} \int_I \left| \frac{a_n(x)\, b_n(t) - a_n(t)\, b_n(x)}{t-x} \right| \frac{dt}{u(t)} \,.
\end{aligned}$$

All integrals over J can be estimated with the help of Lemma 3.10 (applied to the characteristic function g of J). Taking into account that $\int_J |t-x|^{-1} dt \le c \ln n$ we obtain

$$\begin{aligned}
&\left|I_{\alpha,\beta}\left([h(x,\,.\,) - h_n(x,\,.\,)] f_n\right|\right. \\
&\le c\, \|f_n\|_{u(\alpha,\beta)} \left(\frac{(v^{\alpha,\beta}u)^{-1}(x)}{n^{\widetilde{\gamma}} \ln^{\widetilde{\delta}} n} + \int_I \left| \frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\, b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\, b(x)}{t-x} \right| \frac{dt}{u(t)} \right. \\
&\qquad \left. + \int_I \left| \frac{a_n(x)\, b_n(t) - a_n(t)\, b_n(x)}{t-x} \right| \frac{dt}{u(t)} \right) .
\end{aligned} \tag{5.88}$$

If we write

$$\begin{aligned}
&(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\, b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\, b(x) \\
&\quad = -(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\, [b(x) - b(t)] \\
&\qquad + b(x) \left[v^{-\alpha,-\beta}(x)\, (\widetilde{a}\, v^{\alpha,\beta} \sigma_{-\alpha,-\beta})(x) - v^{-\alpha,-\beta}(t)\, (\widetilde{a}\, v^{\alpha,\beta} \sigma_{-\alpha,-\beta})(t) \right] \\
&\quad =: -(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\, [h_1(x) - h_2(t)] + b(x) \left[v^{-\alpha,-\beta}(x)\, h_2(x) - v^{-\alpha,-\beta}(t)\, h_2(t) \right],
\end{aligned}$$

then we see that the first integral on the right hand side of (5.88) can be written as a sum, the addends of which can be both estimated as the integral I_1 in the proof of Theorem 3.1.

(Consider I_1 with $v_1 = P_1 = w_1 = 1$ and $v_2 = v^{\alpha+\widetilde{k},\beta+\widetilde{l}}$, $P_2 = v^{\widetilde{k},\widetilde{l}}$, $w_2 = 1$, respectively.) Consequently, by (3.23),

$$\int_I \left| \frac{(\widetilde{a}\,\sigma_{-\alpha,-\beta})(x)\,b(t) - (\widetilde{a}\,\sigma_{-\alpha,-\beta})(t)\,b(x)}{t-x} \right| \frac{dt}{u(t)} \le c\,\frac{(v^{\alpha,\beta}u)^{-1}(x)}{v^{\alpha+2\widetilde{k},\beta+2\widetilde{l}}(x)\,n^{\theta\eta}} \le c\,\frac{(v^{\alpha,\beta}u)^{-1}(x)}{n^{\widetilde{\gamma}}\ln^{\widetilde{\delta}} n}\,,$$

supposed that $\theta > \left(\widetilde{\gamma} + 2\max\left\{\alpha + 2\widetilde{k}, \beta + 2\widetilde{l}\,\right\}\right)/\eta$, where η is a Hölder exponent of b and $\widetilde{a}\,v^{\alpha,\beta}\sigma_{-\alpha,-\beta}$. Here we took into account that, in view of (5.64), $1 \pm x \ge c\,n^{-2}$. To estimate the second integral on the right hand side of (5.88) we use Markov's inequality (1.53) and Lemma 3.10,

$$\int_I \left| \frac{a_n(x)\,b_n(t) - a_n(t)\,b_n(x)}{t-x} \right| \frac{dt}{u(t)} \le (\|a_n'\|\,\|b_n\| + \|a_n\|\,\|b_n'\|) \int_I \frac{dt}{u(t)} \le c\,\frac{\|a_n\|\,\|b_n\|}{u(x)\,n^{\theta-2}}\,.$$

Using $\|b_n\| \le c$, $\|a_n\| \le c\,n^2\|a_n\|_{1,1} \le c\,n^2\|a_n\|_\psi \le c\,n^2$ (see (1.12)), and again $1 \pm x \ge c\,n^{-2}$, we see that also the last expression is bounded by $c\,(v^{\alpha,\beta}u)^{-1}(x)\,n^{-\widetilde{\gamma}}\ln^{-\widetilde{\delta}} n$ if θ is chosen sufficiently large. ∎

5.6.4 Proof of Corollary 5.18

Let u satisfy (5.44) and let $\varepsilon \in \left(0, \gamma - \max_{i=1}^N(1-\tau_i)\right) \setminus \left\{\gamma - \max_{i=1}^N(1-\tau_i)\right\}$. In view of Lemma 3.9 there exist polynomials $p_n(x,t)$ in x,

$$p_n(x,t) = \sum_{k=0}^{n-1} c_k^{(n)}(t)\,x^k\,,$$

with coefficients $c_k^{(n)} \in \mathbf{C}$ such that

$$\left\|\widetilde{u}(x)\,k_1(x,t)\,w(t) - \widetilde{u}(x)\,p_n(x,t)\right\|_{\mathbf{C}([-1,1]^2)} \le c\,n^{\varepsilon-\gamma}\,, \quad n \in \mathbb{N}\,.$$

Particularly, for every $n \in \mathbb{N}$ and all $(x,t) \in \left[(-1,1) \setminus \{x_i\}_{i=1}^N\right] \times (-1,1)$,

$$u(x)\,k_1(x,t)\,w(t) - u(x)\,p_n(x,t) = \sum_{j=0}^{\infty} u(x)\left[p_{2^{j+1}n}(x,t) - p_{2^j n}(x,t)\right].$$

Using $\left\|u(x)\,(p_{2m} - p_m)(x,t)\right\|_{\mathbf{C}([-1,1]^2)} \le c\,m^{\max_{i=1}^N(1-\tau_i)}\left\|\widetilde{u}(x)\,(p_{2m} - p_m)(x,t)\right\|_{\mathbf{C}([-1,1]^2)}$ (see (1.12)) one can show that this series is even convergent in $\mathbf{C}([-1,1]^2)$ (which implies $u(x)\,k_1(x,t)\,w(t) \in \mathbf{C}([-1,1]^2)$), where

$$\left\|u(x)\,k_1(x,t)\,w(t) - u(x)\,p_n(x,t)\right\|_{\mathbf{C}([-1,1]^2)} \le c\,n^{\varepsilon+\max\{1-\tau_1,\ldots,1-\tau_N\}-\gamma}\,, \quad n \in \mathbb{N}\,.$$

Consequently,

$$\sup_{t\in(-1,1)} \left\|k_1(\,.\,,t)\,w(t)\right\|_{u,\gamma-\max\{1-\tau_1,\ldots,1-\tau_N\}-\varepsilon,0} < \infty$$

and Theorems 3.1 and 3.5 imply $\mathcal{K} \in \mathcal{L}\left(\mathbf{C}_{u(\alpha,\beta)}, \mathbf{C}_u^{\gamma-\max\{1-\tau_1,\ldots,1-\tau_N\}-\varepsilon,0}\right)$. Together with the mapping properties (5.38) and (5.35) of $\widehat{\mathcal{A}}$ and $\mathcal{H}$, which also hold with (γ,δ) replaced by $(\gamma - \max\{1-\tau_1,\dots,1-\tau_N\} - \varepsilon, -1)$, we conclude that $I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K}$ is a compact operator in $\mathbf{C}_{u(\alpha,\beta)}$ with

$$\ker(I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K}) \subseteq \begin{cases} \mathbf{C}_{u(\alpha,\beta)}^{\gamma-\max\{1-\tau_1,\dots,1-\tau_N\}-\varepsilon,-1} & \text{if} \quad b(-1)b(1) \neq 0\,, \\ \mathbf{C}_{u(\alpha,\beta)}^{(\gamma-\max\{1-\tau_1,\dots,1-\tau_N\}-\varepsilon)/2,-1} & \text{if} \quad b(-1)b(1) \neq 0\,. \end{cases}$$

If $2\varepsilon \in (0, \widetilde{\gamma}\,) \setminus \{\gamma - 1\}$, then this can be applied to $u_\varepsilon(x) = v^{\tau_{N+1},\tau_0}(x) \prod_{i=1}^N |x - x_i|^{1-\varepsilon}$ and we see that the $\mathbf{C}_{u_\varepsilon(\alpha,\beta)}$–kernel of $I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K}$ is contained in the $\mathbf{C}_{u(\alpha,\beta)}$–kernel of $I + \mathcal{H} + \widehat{\mathcal{A}}\mathcal{K}$ for all u with

$$\max\{1-\tau_1,\dots,1-\tau_N\} + \varepsilon < \widetilde{\gamma} \quad \text{and} \quad \min\{1-\tau_1,\dots,1-\tau_N\} > \varepsilon\,,$$

since $\mathbf{C}_{u_\varepsilon(\alpha,\beta)}^{\widetilde{\gamma}-2\varepsilon,-1} \subseteq \mathbf{C}_{u(\alpha,\beta)}$ in view of assertion (vii) of Theorem 1.11. Thus, for all these u, the $\mathbf{C}_{u(\alpha,\beta)}$–kernel of $I+\mathcal{H}+\widehat{\mathcal{A}}\mathcal{K}$ is equal to the $\mathbf{C}_{u_\varepsilon(\alpha,\beta)}$–kernel of $I+\mathcal{H}+\widehat{\mathcal{A}}\mathcal{K}$. Particularly, if this kernel is trivial and if $g \in \mathbf{C}_{\widetilde{u}}^{\gamma,\delta+1}$, which implies $g \in \mathbf{C}_{u_\varepsilon}^{\gamma-\varepsilon,\delta+1} \subseteq \mathbf{C}_{u_\varepsilon}^{\gamma-2\varepsilon,0}$ and, consequently, $\widehat{\mathcal{A}}\,g \in \mathbf{C}_{u_\varepsilon(\alpha,\beta)}^{\widetilde{\gamma}-2\varepsilon,-1} \subseteq \mathbf{C}_{u(\alpha,\beta)}$ (use again Theorem 1.11 and (5.38)), then the equation (5.14) possesses a unique solution in $\mathbf{C}_{u(\alpha,\beta)}$. Let us show that, in this case, the equations (5.14) are uniquely solvable for all $n \geq n_0$, where

$$\|f^* - f_n^*\|_{u_\varepsilon(\alpha,\beta)} \leq \frac{c}{n^{\widetilde{\gamma}-4\varepsilon}}\, \|g\|_{\widetilde{u},\gamma,\delta+1}\,. \tag{5.89}$$

Here we assume that $4\varepsilon < \widetilde{\gamma}$. All assumptions of Theorem 5.14 are satisfied with u and (γ,δ) replaced by u_ε and $(\gamma-2\varepsilon,-1)$, except the first condition in (5.40) which is replaced by the corresponding part of (5.43). However, the proof of (5.89) remains almost the same as that of the corresponding assertion of Theorem 5.14. Indeed, the only place there we have to modify the proof of Theorem 5.14 is the estimate (5.77) (with u_ε instead of u) in which we get $\widetilde{u}^{\,-1}(x)$ on the right hand side. But this yields only an additional factor n^ε if we replace $\widetilde{u}^{\,-1}(x)$ by $u_\varepsilon^{-1}(x)$, since $\widetilde{u}(x) \geq c\,n^{-\varepsilon}u_\varepsilon(x)$ because of

$$x \in \left\{x_{n(\alpha,\beta),j}^{-\alpha,-\beta}\right\}_{j=1-s}^{n+r} \setminus \left\{x_{n(\alpha,\beta),k(i)}^{-\alpha,-\beta}\right\}_{i=1}^{N}$$

and, consequently, $|x - x_i| \geq \left|x_{n(\alpha,\beta),k(i)}^{-\alpha,-\beta} - x_{n(\alpha,\beta),k(i)\pm1}^{-\alpha,-\beta}\right|/\,2 \geq c\,n^{-1}$, $i = 1,\dots,N$ (see (5.71)). Now, (5.89) is proved and it remains to mention that (5.45) (with 3ε instead of ε) follows from

$$f^* - f_n^* = \sum_{j=0}^{\infty} \left(f_{2^{j+1}n}^* - f_{2^j n}^*\right)$$

because of $\left\|f_{2^{j+1}n}^* - f_{2^j n}^*\right\|_{u(\alpha,\beta)} \leq c\,(2^j n)^{\max\{1-\tau_1,\dots,1-\tau_N\}-\varepsilon}\left\|f_{2^{j+1}n}^* - f_{2^j n}^*\right\|_{u_\varepsilon(\alpha,\beta)}$ (see (1.12)). ■

5.6.5 Proofs of Lemmas 5.20, 5.21 and 5.23

We need the equation

$$A_{-\alpha,-\beta}\, p_m^{-\alpha,-\beta} = \operatorname{sign}(\alpha)\, p_n^{\alpha,\beta} \quad (n = m + \widetilde{k} + \widetilde{l} - 1)\,, \tag{5.90}$$

which holds for all $m \in \mathbb{N} \cup \{0\}$ (see [PS, Theorems 9.9 and 9.14]).

Proof of Lemma 5.20. It is known (see, e.g., [PS, Corollary 9.20 and its proof]) that, for all $p \in \Pi_{2m+1}$,

$$(A_{-\alpha,-\beta}\, p)(x) = \operatorname{sign}(\alpha)\, p(x) \frac{p_n^{\alpha,\beta}(x)}{p_m^{-\alpha,-\beta}(x)} + \sum_{k=1}^{m} \mu_k \frac{p(y_k)}{y_k - x}\,. \tag{5.91}$$

(Use $Q_m^{-\alpha,-\beta}$ to compute the integral on the right side of the equation $(A_{-\alpha,-\beta}\, p)(x) = p(x)(A_{-\alpha,-\beta}1)(x) + \pi^{-1}\int_{-1}^{1}(t-x)^{-1}(p(t)-p(x))\sigma_{-\alpha,-\beta}(t)\,dt$. The resulting equation applied to $p = p_m^{-\alpha,-\beta}$ yields a formula for $(A_{-\alpha,-\beta}1)(x)$ because of (5.90).) From equations (5.91) and (5.31) we obtain the assertion. ■

Proof of Lemma 5.21. Let $k \in \mathbb{N}$ and set $l = k + \widetilde{k} + \widetilde{l} - 1$. Then, by (5.90) and (5.50),

$$\begin{aligned}
\operatorname{sign}(\alpha)\, \beta_{k+1}^{-\alpha,-\beta} p_{l+1}^{\alpha,\beta} &= A_{-\alpha,-\beta}\big(\beta_{k+1}^{-\alpha,-\beta} p_{k+1}^{-\alpha,-\beta}\big) \\
&= A_{-\alpha,-\beta}\big((t - \alpha_k^{-\alpha,-\beta})\, p_k^{-\alpha,-\beta} - \beta_k^{-\alpha,-\beta} p_{k-1}^{-\alpha,-\beta}\big) \\
&= A_{-\alpha,-\beta}\big(t\, p_k^{-\alpha,-\beta}\big) - \operatorname{sign}(\alpha)\big[\alpha_k^{-\alpha,-\beta} p_l^{\alpha,\beta} + \beta_k^{-\alpha,-\beta} p_{l-1}^{\alpha,\beta}\big]\,.
\end{aligned}$$

Moreover, $A_{-\alpha,-\beta}\big(t\, p_k^{-\alpha,-\beta}\big) - x\, A_{-\alpha,-\beta}\, p_k^{-\alpha,-\beta} = \pi^{-1}\int_{-1}^{1} p_k^{-\alpha,-\beta}(t)\,\sigma_{-\alpha,-\beta}(t)\,dt = 0$, i.e., $A_{-\alpha,-\beta}\big(t\, p_k^{-\alpha,-\beta}\big) = \operatorname{sign}(\alpha)\, x\, p_l^{\alpha,\beta}$ in view of (5.90). Consequently,

$$\beta_{k+1}^{-\alpha,-\beta} p_{l+1}^{\alpha,\beta}(x) = \big(x - \alpha_k^{-\alpha,-\beta}\big)\, p_l^{\alpha,\beta}(x) - \beta_k^{-\alpha,-\beta} p_{l-1}^{\alpha,\beta}(x)\,.$$

If we compare the coefficients of x^{l+1}, then we obtain $\beta_{k+1}^{-\alpha,-\beta} = \beta_{l+1}^{\alpha,\beta}$. Now we look at the coefficients of x^l and we conclude $\alpha_k^{-\alpha,-\beta} = \alpha_l^{\alpha,\beta}$. Now it is clear that $\beta_k^{-\alpha,-\beta}$ must be equal to $\beta_l^{\alpha,\beta}$ if $l \geq 1$. ■

Proof of Lemma 5.23. Let $p = p_n^{\alpha,\beta}$ and $q = p_m^{-\alpha,-\beta}$. If $j \in \{1, \ldots, m\}$, then

$$l_j^L(x) = \frac{q(x)}{q'(y_j)(x - y_j)} \prod_{i \in \{1-s, m+r\} \cap \{0, m+1\}} \frac{x - y_i}{y_j - y_i} \prod_{i=1}^{N} \frac{y_j - y_{k(i)}}{x - y_{k(i)}}\,.$$

Moreover, by (5.90) and (5.18),

$$\begin{aligned}
\operatorname{sign}(\alpha)\, p(y_j) &= (A_{-\alpha,-\beta}\, q)(y_j) = \frac{1}{\pi}\int_{-1}^{1} \frac{q(t)}{t - y_j}\, \sigma_{-\alpha,-\beta}(t)\,dt \\
&= \sum_{k=1}^{m} \mu_k \frac{q(t)}{t - y_j}_{|t=y_k} = \mu_j\, q'(y_j)\,.
\end{aligned}$$

Thus, the first assertion of Lemma 5.23 is proved. The formulas for $L_{0,k}$ and $L_{m+1,k}$ $(k=1,\dots,n)$ are consequences of

$$\begin{aligned} l_0^L(x) &= \frac{q(x)}{q_0} \prod_{i\in\{m+r\}\cap\{m+1\}} \frac{x-y_i}{y_0-y_i} \prod_{i=1}^{N} \frac{y_0-y_{k(i)}}{x-y_{k(i)}}\,, \\ l_{m+1}^L(x) &= \frac{q(x)}{q_{n+1}} \prod_{i\in\{1-s\}\cap\{0\}} \frac{x-y_i}{y_{m+1}-y_i} \prod_{i=1}^{N} \frac{y_{m+1}-y_{k(i)}}{x-y_{k(i)}}\,. \end{aligned}$$

From the above formulas for $l_k^L(x)$, $k\in\{1-s,\dots,m+r\}\setminus\{k(1),\dots,k(N)\}$, we obtain $L_{k,n+i}$ if we consider the limit $x\to y_{k(i)}$. (Here we use again $\mu_j\, q'(y_j)=\operatorname{sign}(\alpha)\, p(y_j)$.) The formulas for $P_{k,i}$ are obtained from those for $L_{k,n+i}$ by changing $m\leftrightarrow n$, $-\alpha\leftrightarrow\alpha$, $-\beta\leftrightarrow\beta$.

■

5.7 Notes and Comments

5.1. Propositions 5.1–5.6 are known if spectral methods and Jacobi weights u,w are considered (see [JL1]), where in such a case the initial equation together with some additional conditions (compare Remark 5.17 for the case of operators A with constant coefficients) is even equivalent to the regularized equation, supposed that $\varkappa=\varkappa_0-k-l\geq 0$ for k,l defined by (2.54). More details with respect to the connection between initial equation and regularized equation in the case of spectral methods are already given in Section 4.4.

5.2. Quadrature methods using Gaussian quadrature rules for the approximation of weakly singular integral operators and zeros of orthogonal polynomials as collocation points are well studied in the case of spectral methods (see the references given in the introduction). The non-spectral methods which can be found in the literature are different from that considered in Section 5.2 (compare Section 2.6). The idea of defining interpolation operators $L_{n,u}^{\alpha,\beta}$ simply by omitting "critical" knots is not new. For example, in [MM] one can find results on corresponding modified interpolation operators based on zeros of Jacobi polynomials. But in [MM] no additional knots $x_{n,0}^{\alpha,\beta}$ and $x_{n,n+1}^{\alpha,\beta}$ are considered and the result on the $\mathcal{L}(\mathbf{C}_u)$–norms of the interpolation operators (see [MM, Theorem 3.3]) is only applicable in cases where $\tau_1,\dots,\tau_N\geq 1$. For this reason we have proved a corresponding result in Subsection 5.6.3 (see (5.62)). We mention that a better result can be proved for certain modified interpolation operators based on the zeros of orthogonal polynomials with respect to weights having zeros inside $(-1,1)$ (see [MM]). But we have not used such interpolation operators, since this would yield too many complications in the theory of the quadrature method and in its numerical implementation.

5.3 The results of Section 5.3 are new. Only in the situation considered in Remark 5.17 our quadrature method (with the slight modification from Remark 5.17) is the same than that considered, e.g., in [JL2]. But also in this special situation result on the weighted uniform convergence can be found only for the case of Jacobi weights u in the literature.

5.4 Our derivation of the linear system is based on well-known results from the theory of spectral methods for Cauchy singular integral equations on $[-1,1]$ (see [PS, Chapter 9]). The principle of the Stieltjes procedure given in Subsection 5.4.2 is taken from [G].

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